MATH10222, Chapter 1: Introduction

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Last modified: 2017-03-17

FAQ

1. Can I ignore these online notes?

   If you attend all the lectures and problem/feedback classes, then you have all the material and these notes are not needed.

2. If I have these notes, can I skip the classes?

   I advise against it. My past experience is that students that do not attend the lectures/supervisions can often perform poorly, even if they obtain the material that was delivered in the classes.

3. How does the material in these notes differ from that delivered in the lectures?

   When writing on the blackboard I will de-emphasise the written text and emphasise the detail and reasoning behind the mathematical steps. In these notes I reverse this approach, being a little more discursive with less of the algebra shown. There will be some parts of the online notes where I will include additional material that will not be discussed (in much detail) in the lectures; this will be clearly noted by 'extra' in the margin.

4. Is the sequence/numbering consistent between the online notes and the lectures?

   Yes, the sections and chapters are all consistent (individual equation numbers are not).

5. Can I ignore any material marked as 'extra' if I use the online notes when revising for the final exam?

   Yes, but looking through the extra material may still help your understanding of other parts of the course.

6. Do I have to memorise a lot of formulae?

   No. This course is not testing how good your memory is. Any expressions/formulae that I expect you to have memorised for the examination are highlighted with a surrounding box – this is a very small number and they should be entirely obvious by the time you have worked through all the problems.

Obviously there are many 'applications' of calculus, however in this course we will concentrate specifically on the problems that primarily gave rise to the development of calculus. Our emphasis will be on problems of 'mechanics', and more specifically we will mostly concentrate on single-'particle' dynamics. However, by the end of this course we should hopefully have covered some of the bigger topics of pre-20^{th} century science.

1 Introduction.

This course makes no assumptions that you have taken any previous 'mechanics' course before. As such, we'll start with some very basic definitions that many will already be familiar with:

- A **particle** has a position in space and a **mass** (see below), but it does not have any size or shape. It is a single point in space; a point-mass.

- In this course we will spend most of our time discussing the motion of a particle (or sometimes, a system of particles) and will not discuss the motion of bodies of finite size. Since a particle consists of a single point in space we can describe its path, as time varies, using a single vector that is measured relative to some fixed origin of a coordinate system. Throughout this course we will use $\underline{r}(t)$ to denote the **position** of a particle relative to an origin at a given time t. Let's assume for the moment that we can pick such an 'origin', but we will return to reconsider this issue towards the end of the course.

  In a Cartesian coordinate system relative to a fixed origin O, with unit basis vectors $\underline{i}, \underline{j}, \underline{k}$, the **position** vector can be written in the form

  $$\boxed{\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}\,;}$$

  where $x(t), y(t), z(t)$ are the three coordinates needed to identify the particle in a three-dimensional space at any given value of time t. Note that the **distance** of the particle from O is then defined to be the scalar quantity $|\underline{r}|$.

  You MUST always distinguish between vectors and scalars in your written work. You will definitely lose marks in any examination for failing to keep a consistent notation. I will actively deduct marks from anyone that not only forgets to distinguish vectors from scalars, but goes one step further and incorrectly treats vectors as though they are scalars. **Note**

- The **velocity** of a particle with position vector $\underline{r}(t)$ is defined to be the vector

  $$\boxed{\underline{v}(t) = \dot{\underline{r}}(t)\,.}$$

  In component form relative to Cartesian coordinate system this is

  $$\dot{\underline{r}}(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}\,.$$

  Velocity is measured in meters per second (m/s). The **speed** of a particle is a scalar quantity, defined to be the magnitude of the velocity, that is $|\underline{v}|$.

- The **acceleration** of a particle with position vector $\underline{r}(t)$ is defined to be the vector

  $$\boxed{\underline{a}(t) = \dot{\underline{v}}(t) = \ddot{\underline{r}}(t)\,.}$$

  In component form relative to a Cartesian coordinate system this is

  $$\ddot{\underline{r}}(t) = \ddot{x}(t)\underline{i} + \ddot{y}(t)\underline{j} + \ddot{z}(t)\underline{k}\,.$$

  Acceleration is measured in meters per second per second (m/s^2).

- A vector quantity that will be important later is the mass times the vector velocity

$$\underline{p} = m\underline{v} \, ,$$

for a particle of mass m. This is defined to be the **linear momentum** of the particle.

- **Mass** is an intrinsic property of matter (measured in kilograms, kg) related to a particle's reluctance to being accelerated. This is a rather vague definition of mass, and for good reason, the issue is more subtle than you might think.

Extra

Our basis for the concept of mass arises from an empirical law[1], the 'law of mutual interaction'.

Suppose that two particles P_1 and P_2 interact only with eachother (by some unspecified mechanism), such that P_1 induces an acceleration $\underline{a}_{21}$ in P_2 and P_2 induces an acceleration $\underline{a}_{12}$ in P_1. Then the accelerations $\underline{a}_{21}$ and $\underline{a}_{12}$ are opposite in direction and parallel to the line joining P_1 and P_2. Furthermore, the ratio of $|\underline{a}_{21}|/|\underline{a}_{12}|$ is a constant that is independent of the nature of the interaction. In other words, it doesn't matter what the mechanism of attraction is (a spring, electric charge or even a bit of elastic) the ratio of the two accelerations is the same.

This observation allows us to define a useful property of the $P_{1,2}$ system that is independent of what causes the accelerations or how the particles are moving. We call this property the (inertial) mass ratio of the two particles:

$$\frac{m_1}{m_2} = \frac{|\underline{a}_{21}|}{|\underline{a}_{12}|} \, .$$

If we can define a mass ratio for any pair of particles, then we can define the (inertial) **mass** for any individual particle by simply choosing one object as the reference mass. Our reference mass is actually a block of platinum-iridium stored at the International Bureau of Weights and Measures in France. It is this reference block that is the 'kilogram'.

This is not quite the full story. There is an extra requirement that our definition of mass must be consistent if we now introduce an intermediate third particle. For a more detailed discussion see the book 'Classical Mechanics' by R.D. Gregory, p.55.

- It is worth briefly leaping ahead of schedule here to note that **weight** is commonly confused with **mass** but is a 'force' (measured in Newtons, N) caused by the gravitational attraction of a body (eg. you) to some other typically larger body (eg. the Earth). We have not defined what a 'force' is yet, again the issue is more subtle than you might think. Similarly we postpone a discussion of gravitational attraction until the end of this chapter.

[1] that is, something that is held to be always true on the basis of all known experiments/observations; it is not provable in a mathematical sense.

Example 1.1: Motion under constant acceleration

In a Cartesian coordinate system with basis vectors $\{\underline{i}, \underline{j}, \underline{k}\}$, a particle P is at the origin $\underline{r} = \underline{0}$ at time $t = 0$ and has an initial velocity $U\underline{i}$. P is subjected to an acceleration $\underline{a}(t) = a\underline{i}$ for $t \geq 0$, where a is a constant.

Question: Find the position of P at time t.

Answer: The question is asking us to find $\underline{r}(t)$; the position vector of P. We are told the acceleration, and we know the connection between acceleration and position is

$$\underline{a}(t) = \ddot{\underline{r}}(t) \, .$$

So the question, despite it's mechanics context, is really about solving a *very* simple ODE:

$$\ddot{\underline{r}}(t) = a\underline{i} \, .$$

This is a second-order, constant coefficient, ODE and therefore requires two initial conditions. The question provides the two conditions:

$$\underline{r}(t = 0) = \underline{0} \, , \quad \dot{\underline{r}}(t = 0) = U\underline{i} \, .$$

There are a number of ways one can solve this simple system. The obvious way is to just integrate twice, however this is a vector equation whereas all the examples you have seen from the first half of the course were scalar equations. Fortunately this is not a problem.

<u>Method 1</u>

The long way of solving this would be to use a component form for the position vector

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k} \, ,$$

then we obtain three scalar ODEs on taking each component separately:

$$\ddot{x} = a \, , \quad \ddot{y} = 0 \, , \quad \ddot{z} = 0 \, .$$

Integrating once

$$\dot{x} = at + c_1 \, , \quad \dot{y} = c_2 \, , \quad \dot{z} = c_3 \, ,$$

where $c_{1,2,3}$ are constants of integration. We know the initial velocity is

$$\dot{\underline{r}}(t = 0) = \dot{x}(0)\underline{i} + \dot{y}(0)\underline{j} + \dot{z}(0)\underline{k} = U\underline{i} \, ,$$

which shows that $c_1 = U$, $c_2 = 0$, $c_3 = 0$.

Integrating again

$$x = \frac{a}{2}t^2 + Ut + d_1 \, , \quad y = d_2 \, , \quad z = d_3 \, ,$$

where $d_{1,2,3}$ are constants of integration. We know the initial position is

$$\underline{r}(t = 0) = x(0)\underline{i} + y(0)\underline{j} + z(0)\underline{k} \, ,$$

which shows that $d_{1,2,3} = 0$.

So the position of P is

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k} = \left(\frac{a}{2}t^2 + Ut\right)\underline{i}\,.$$

Method 2

The above method is rather long-winded. Rather than splitting into component parts, we can just treat the vector ODE in the obvious manner. Starting from

$$\ddot{\underline{r}}(t) = a\underline{i}\,,$$

we integrate once to find

$$\dot{\underline{r}}(t) = at\underline{i} + \underline{c}\,,$$

where the only difference now is that $\underline{c}$ is a constant *vector* of integration. Using the initial condition $\dot{\underline{r}}(0) = U\underline{i}$ shows that

$$\underline{c} = U\underline{i}\,.$$

Integrating again:

$$\underline{r}(t) = \frac{a}{2}t^2\underline{i} + Ut\underline{i} + \underline{d}\,,$$

where $\underline{d}$ is a constant *vector* of integration. Again using the initial condition $\underline{r}(0) = \underline{0}$ shows that $\underline{d} = \underline{0}$, so the position of P is

$$\underline{r}(t) = \left(\frac{a}{2}t^2 + Ut\right)\underline{i}\,,$$

as before.

Note

If you have done a mechanics course before, you may recognise the solutions above as being a vector form of one of the 'SUVAT' equations for particle motion under constant acceleration. Please do NOT memorise the 'SUVAT' equations, they only apply to this special case and as you can see integrating twice is hardly a challenge. You should prioritise understanding rather than attempting to memorise material; this is not a course about committing random things to memory.

2 Forces

For a particle P, we will define the **force** acting on P to be the product of the mass of P (as defined above) and the vector acceleration of P. The force is therefore a vector (that is, a quantity with direction and magnitude) whose line of action is through the point P.

We will treat forces as vectors, with the usual associated definitions:

- The **resultant** of a system of forces is just their vector sum.
- Like any other vector, we can decompose forces into **component** parts.
- Like any other vector, the **magnitude** of a force $\underline{F}$ is the scalar quantity $|\underline{F}|$.

It is very important that you are comfortable in using vectors. Now is a good time to review your notes from semester 1.

Extra

We have side-stepped some rather subtle issues here. We have defined the concept of a force using Newton's second law (which you saw briefly in the first half of the course and we will meet again in chapter 2). However this is not sufficient for us to treat them as vectors, because vectors also have other algebraic properties (that is, we have rules for vector algebra).

There is however another empirical law, the 'law of multiple interactions'. This law is based upon the observation that, if particles $P_0, ..., P_N$ are interacting only with eachother, then the acceleration of P_0 is

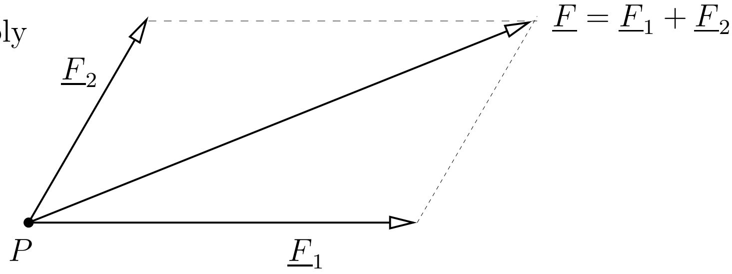
$$\underline{a}_0 = \underline{a}_{01} + \underline{a}_{02} + \cdots + \underline{a}_{0N} \, ,$$

where $\underline{a}_{0i}$ is the acceleration induced in P_0 by its pairwise interaction with P_i for $i = 1, ..., N$. In other words, the acceleration induced in P_0 is observed to be precisely the vector sum of the individual accelerations that arise by considering each particle individually. If we can sum the accelerations using the usual rules for vector addition, then we can similarly sum forces using the definition above.

Example 2.1: The resultant of two forces as a vector sum

Question: A particle P is acted upon by two co-planar forces $\underline{F}_1$ and $\underline{F}_2$ (which are not co-linear). Sketch a diagram showing the resultant force, $\underline{F}$, acting on P.

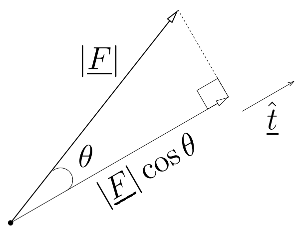
Answer: The **resultant** is defined to be simply the linear sum $\underline{F} = \underline{F}_1 + \underline{F}_2$, so the diagram is:



Example 2.2: Finding the component of a force in a particular direction

Question: Find the component of the force $\underline{F} = (3, 2, -1)$ in the direction of the unit vector $\hat{\underline{t}} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$.

Answer: The **component** of a force $\underline{F}$ in a direction $\hat{\underline{t}}$ can be found by drawing a force triangle, or by using the dot product.



$\underline{F}_{\hat{t}} = \underline{F} \cdot \hat{\underline{t}} = |\underline{F}| \cos\theta$,
where θ is the angle between $\underline{F}$ and $\hat{\underline{t}}$, and $\hat{\underline{t}}$ is a unit vector.

Note that the dot product will only provide the component if $\hat{\underline{t}}$ (in this notation) is a *unit* vector, that is, $|\hat{\underline{t}}| = 1$.

For this question we do not know θ (the angle between $\underline{F}$ and $\hat{\underline{t}}$), so the most convenient method is to use the dot product:

$$\underline{F}_{\hat{t}} = (3, 2, -1) \cdot (1/\sqrt{2}, -1/\sqrt{2}, 0) = \frac{1}{\sqrt{2}}(3-2) = \frac{1}{\sqrt{2}}\,.$$

Example 2.3: Finding the components of the weight of a particle sitting on a slope

Question: A particle P rests on a plane that is at angle α to the horizontal (the direction given by the unit vector $\underline{i}$) and experiences a force of magnitude W downwards (the direction given by the unit vector $-\underline{j}$). Resolve the force into components parallel and perpendicular to the plane.

Answer: In any of these types of questions, the first thing to do is to draw a sketch of the problem, see Figure 1.

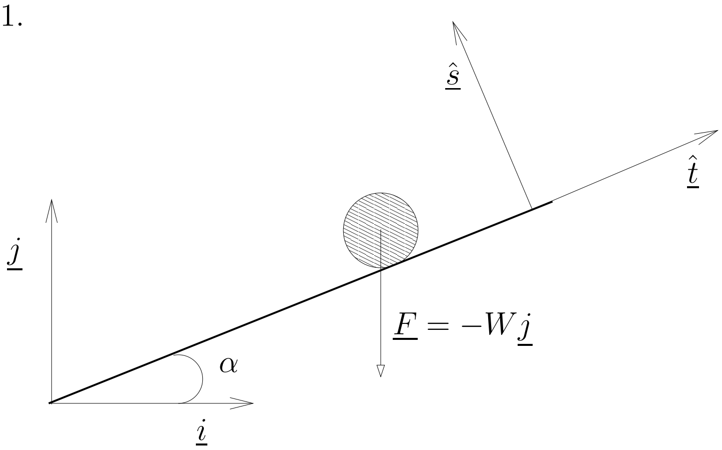


Figure 1: A particle rests on a plane inclined at an angle α to the horizontal and experiences a downward force of magnitude W. The unit vectors normal and tangential to the plane are denoted by $\hat{\underline{s}}$ and $\hat{\underline{t}}$, respectively.

The straightforward method is to find expressions for vectors parallel and perpendicular to the plane in terms of the Cartesian base vectors and then take dot products. We have

$$\hat{\underline{t}} = \cos\alpha\,\underline{i} + \sin\alpha\,\underline{j} \quad \text{and} \quad \hat{\underline{s}} = -\sin\alpha\,\underline{i} + \cos\alpha\,\underline{j}.$$

The components of the force parallel and perpendicular to the plane are then

$$\underline{F} \cdot \hat{\underline{t}} = \begin{pmatrix} 0 \\ -W \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = -W\sin\alpha\,,$$

and

$$\underline{F} \cdot \hat{\underline{s}} = \begin{pmatrix} 0 \\ -W \end{pmatrix} \cdot \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} = -W\cos\alpha.$$

Thus,

$$\underline{F} = -W\sin\alpha\,\hat{\underline{t}} - W\cos\alpha\,\hat{\underline{s}}\,.$$

An alternative approach is to draw a force triangle and decompose $\underline{F}$ into the perpendicular and parallel components directly.

3 Moments of forces

If a force $\underline{F}$ acts through a point P with position vector $\underline{r}$ (relative to origin O) then we define the **moment** about O to be the quantity

$$\boxed{\underline{L}_O = \underline{r} \wedge \underline{F}\,.}$$

Note

- The subscript O indicates that $\underline{r}$ is the position relative to the origin O, that is, it is the 'moment about the point O'.
- The $\underline{L}_O$ notation used for the moment of a force varies across textbooks, bear this in mind if you are looking for discussions of this topic elsewhere.
- You may have seen moments discussed before, but as a scalar that is 'force times perpendicular distance'. The vector definition here is consistent with that scalar definition.
- Some textbooks may use the phrase 'torque' rather than 'moment'.

If we have N particles, with position vectors $\underline{r}_i$, each acted upon by a different force, $\underline{F}_i$, then the **total moment** for the system (about O) is defined to be the vector sum of the individual moments for each particle:

$$\underline{L}_O = \sum_{i=1}^{i=N} \underline{r}_i \wedge \underline{F}_i\,.$$

Example 3.1: Computing the moment of a force

Question: The force $\underline{F} = 4\underline{j}$ acts through the point $\underline{r} = 3\underline{i}$ in a Cartesian coordinate system with unit basis vectors $\{\underline{i}, \underline{j}, \underline{k}\}$. Find the moment of the force about the origin.

Answer: By definition the moment is

$$\underline{L}_O = \underline{r} \wedge \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix} = 12\underline{k}\,.$$

4 Systems in equilibrium

If a system of particles is each acted upon by a force, then we say that the system is in **equilibrium** if and only if two conditions are met:

1. The resultant (ie. the vector sum) of all the forces must be zero.
2. The total moment about any point in space must be zero.

If we define the 'any point' as A, which has a position vector of $\underline{A}$ relative to the origin O, then the moment of the system about A is

$$\underline{L}_A = \sum_{i=1}^{i=N} (\underline{r}_i - \underline{A}) \wedge \underline{F}_i \,.$$

However, we do not need to test every such $\underline{A}$, because:

$$\begin{aligned} \underline{L}_A &= \sum_{i=1}^{i=N} \underline{r}_i \wedge \underline{F}_i - \sum_{i=1}^{i=N} \underline{A} \wedge \underline{F}_i = \sum_{i=1}^{i=N} \underline{r}_i \wedge \underline{F}_i - \underline{A} \wedge \sum_{i=1}^{i=N} \underline{F}_i \,, \\ &= \sum_{i=1}^{i=N} \underline{r}_i \wedge \underline{F}_i = \underline{L}_O \end{aligned}$$

provided that the resultant of the forces is zero. In other words, to test for equilibrium, we first find the resultant force, if that is zero, then we can test the moment about any convenient point in space. If that moment is zero then all moments are also zero and the system is in equilibrium.

5 Other coordinate systems: polar basis vectors

We will use the polar basis vectors discussed here in later sections of the course, so make sure you understand the notation/concepts in the first supervision class.

If we can describe the motion of a particle in a plane using a Cartesian coordinate system (relative to an origin O) with (unit) basis vectors $\underline{i}, \underline{j}$ then life is fairly simple because $\underline{i}, \underline{j}$ are constants. Therefore if we have a particle P with a position vector of $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$, then differentiating to find the velocity is easy because

$$\underline{v}(t) = \frac{\mathrm{d}\underline{r}(t)}{\mathrm{d}t} = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} \,.$$

This just follows from the product rule. For example for the first component:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)\underline{i}) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\underline{i} + x(t)\frac{\mathrm{d}\underline{i}}{\mathrm{d}t} = \dot{x}(t)\underline{i} \,,$$

because

$$\frac{\mathrm{d}\underline{i}}{\mathrm{d}t} = 0 \,,$$

as $\underline{i}$ is independent of time. Similarly for the second component.

However, in some problems (eg. planetary motion) we prefer to use the (unit) polar basis vectors $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ (see figure 2). In this coordinate system the position vector of P is

$$\boxed{\underline{r} = r\hat{\underline{r}} \,.}$$

You may find this notation confusing at first, but remember: **Note**

- $\underline{r}$ is the position vector of P relative to O, just as before.

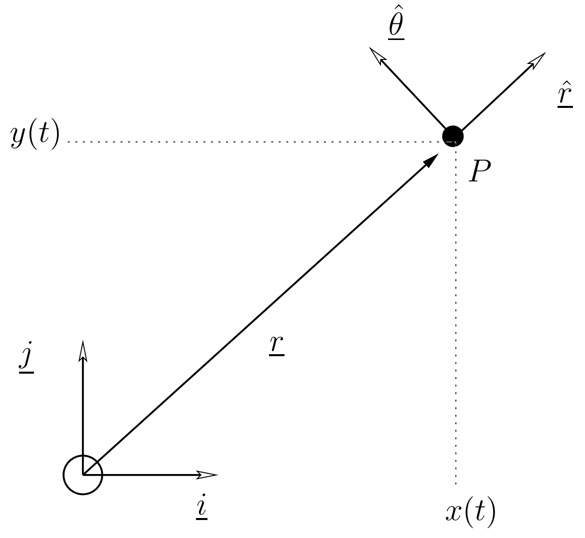


Figure 2: Using polar basis vectors to describe the motion of a particle P in a plane, relative to an origin O. In some cases it is more convenient to use $\underline{r} = r\hat{\underline{r}}$ where $r = |\underline{r}|$ instead of the Cartesian form $\underline{r} = x\underline{i} + y\underline{j}$. Although this is often more convenient, there is a penalty we have to pay when it comes to evaluating the velocity or acceleration; they are made more complicated by the fact that now $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ change as P moves in the plane.

- $\hat{\underline{r}}$ is a **unit** vector in the direction of OP. (We use the hat notation to emphasise that these are unit vectors).
- r is a scalar, and is simply how far away from O the particle P is. Because $\hat{\underline{r}}$ is a unit vector (i.e., $|\hat{\underline{r}}| = 1$), $|\underline{r}| = r$.

At first sight this may look like you are defining a point in a plane using just one scalar coordinate, r, however this is not true, as there is a θ coordinate that is being implicitly used to define the basis vector $\hat{\underline{r}}$. The unit vector $\hat{\underline{r}}$ always points in the direction of OP, so as P moves with time, so does $\hat{\underline{r}}$.

This coordinate system makes life easier in some respects, but the penalty we have to pay is that

$$\frac{\mathrm{d}\underline{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r(t)\hat{\underline{r}}) = \dot{r}(t)\hat{\underline{r}} + r(t)\frac{\mathrm{d}\hat{\underline{r}}}{\mathrm{d}t},$$

where now $\hat{\underline{r}}$ varies as P moves, so the last term in this expression no longer vanishes as it did in the Cartesian case. However, we can determine this last term (as well as the corresponding term for $\hat{\underline{\theta}}$) by noting that

$$\begin{aligned} \hat{\underline{r}} &= \cos\theta\,\underline{i} + \sin\theta\,\underline{j}, \\ \hat{\underline{\theta}} &= -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}, \end{aligned}$$

so we can differentiate using the chain rule:

$$\begin{aligned} \frac{\mathrm{d}\hat{\underline{r}}}{\mathrm{d}t} &= \frac{\mathrm{d}\hat{\underline{r}}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = (-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})\,\dot{\theta} = \dot{\theta}\hat{\underline{\theta}}, \\ \frac{\mathrm{d}\hat{\underline{\theta}}}{\mathrm{d}t} &= \frac{\mathrm{d}\hat{\underline{\theta}}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = (-\cos\theta\,\underline{i} - \sin\theta\,\underline{j})\,\dot{\theta} = -\dot{\theta}\hat{\underline{r}}. \end{aligned}$$

The resulting expression for the velocity is therefore

$$\underline{v} = \dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}\,, \tag{1}$$

and repeating this process (see Example sheet 1) provides an expression for the acceleration

$$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{\hat{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{\hat{\theta}}\,. \tag{2}$$

Example 5.1: Uniform circular motion

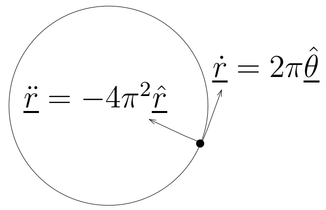
Question: A particle P moves anticlockwise around a circle of unit radius at constant speed such that it performs one complete revolution every second. Find the acceleration of P.

Answer: We choose to represent the motion of the particle in plane polar coordinates with an origin at the centre of the particle's circular path. The position of the particle is given by $\underline{r} = r\,\underline{\hat{r}}$, where $r = 1$. The particle turns through an angle of 2π radians every second, so $\theta = 2\pi t$ and

$$r = 1 \quad\Rightarrow\quad \dot{r} = \ddot{r} = 0 \quad \text{and} \quad \theta = 2\pi t \quad\Rightarrow\quad \dot{\theta} = 2\pi \quad\Rightarrow\quad \ddot{\theta} = 0.$$

Using equation (2) the acceleration is given by

$$\underline{a} = -r\,(2\pi)^2\,\underline{\hat{r}} = -4\pi^2\underline{\hat{r}},$$



which implies that uniform circular motion at constant speed leads to a constant acceleration towards the centre of rotation.

6 Modelling assumptions

Often we want a particle to be geometrically constrained to lie on a slope/table, be connected to a point by a string, move along a wire etc. To achieve this we impose a **force of constraint**. The size and direction of this force is whatever it needs to be to impose the constraint. For example, for a particle that is stationary on a rigid table, we need a force of constraint equal and opposite to the particles weight in order to keep it at the same level (ie. on the table). The force in this case is called a **normal reaction force**.

Example 6.1: A particle at rest on a horizontal plane

Question: A particle of weight $-W\underline{j}$ is in equilibrium on a horizontal plane. What is the normal reaction force?

Answer: For the particle to be in equilibrium we need a zero resultant force, in which case there must be a normal reaction force $N = +W\underline{j}$, as shown in figure 3(a).

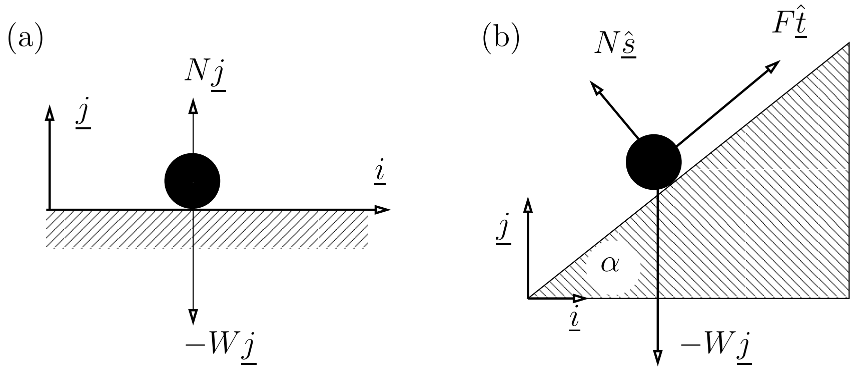


Figure 3: (a) A particle of weight $-W\underline{j}$ resting on a horizontal plane. (b) A particle of weight $-W\underline{j}$ resting on a sloped plane (of angle α to the horizontal), with a frictional force.

Example 6.2: A particle at rest on a sloped plane with friction

Question: A particle of weight $-W\underline{j}$ is in equilibrium on a frictional plane that is at an angle of α to the horizontal. Find the normal reaction force, $N\underline{\hat{s}}$, and the tangential frictional force, $F\underline{\hat{t}}$, as shown in figure 3(b).

Answer: To compare coefficients of the vector forces we can either rewrite $\underline{\hat{s}}$ and $\underline{\hat{t}}$ in terms of $\underline{i}$ and $\underline{j}$ or vice versa. Let's choose to decompose the weight into components parallel and perpendicular to the plane:

$$-W\underline{j} = -W \sin\alpha\, \underline{\hat{t}} - W\cos\alpha\, \underline{\hat{s}}\,.$$

Then for the particle to be in equilibrium we need the resultant force to be zero

$$N\underline{\hat{s}} + F\underline{\hat{t}} - W\underline{j} = N\underline{\hat{s}} + F\underline{\hat{t}} - W\sin\alpha\,\underline{\hat{t}} - W\cos\alpha\,\underline{\hat{s}} = \underline{0}\,.$$

So

$$N = W\cos\alpha\,, \quad F = W\sin\alpha\,.$$

Note that the moment is trivially zero since all forces are acting through the same point.

7 Gravitational fields

Newton's law of gravitational attraction (see figure 4) states that, for a pair of particles of mass m_1 and m_2, the force on mass m_2 due to m_1 is

$$\boxed{\underline{F} = -\frac{Gm_1m_2}{r^2}\underline{\hat{r}}\,,}$$

where the position of m_2 relative to m_1 is $\underline{r} = r\underline{\hat{r}}$ in the usual notation. Here G is a constant of proportionality called the "universal gravitational constant". Clearly for the force on m_1 due to m_2 we have the same magnitude, but in the opposite direction (as we simply reverse $\underline{\hat{r}}$).

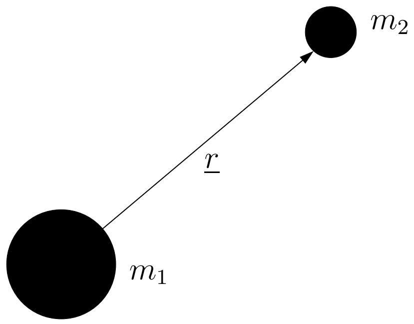


Figure 4: Newton's law of gravitational attraction tells us what the attractive force is between two particles of mass m_1 and m_2, where the position vector of mass m_2 relative to m_1 is $\underline{r}$.

Uniform local gravity

If r does not vary significantly during a particle's motion, then we can approximate the terms Gm_1/r^2 as a single constant.

For example, suppose we have a particle P of mass $m_2 = m$ in the gravitational field of the Earth, whose mass we will denote by $m_1 = E$. Furthermore, suppose the distance of P from the centre of the Earth is $r = R + h$, where R is the radius of the Earth and h is the height of the particle above the surface of the Earth. From Newton's law of gravitational attraction, the magnitude of the attractive force can be approximated if h is much smaller than R:

$$\frac{GEm}{(R+h)^2} \approx \frac{GE}{R^2} m = mg \,,$$

if we define the constant $g = GE/R^2$. We then say that P moves in a 'uniform gravitational field', where for terrestrial problems we can evaluate $g \approx 9.81 m/s^2$.

Suppose we have a system of particles P_i with masses m_i and positions $\underline{r}_i$, where $i = 1, 2, \cdots, N$ in a uniform gravitational field of strength g. As the gravitational field is uniform each particle experiences a force $-m_i g \underline{k}$, where $\underline{k}$ is the upwards pointing unit vector. Then the resultant force for the system is

$$\underline{F} = \sum_{i=1}^{N} -m_i g \underline{k} = -g \underline{k} \left(\sum_{i=1}^{N} m_i \right) = -Mg\underline{k} \,,$$

where

$$M = \sum_{i=1}^{N} m_i \,,$$

is the total mass of the system. The total moment about O (as defined in section 3) is

$$\underline{L}_O = \sum_{i=1}^{N} \underline{r}_i \wedge \underline{F}_i = \sum_{i=1}^{N} \underline{r}_i \wedge (-m_i g \underline{k}) = -g \left(\sum_{i=1}^{N} m_i \underline{r}_i \wedge \underline{k} \right) \,.$$

Hence we **define** the **centre of mass** of the system to be

$$\underline{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \underline{r}_i \,,$$

and the moment of the system is then

$$\underline{L}_O = -gM\underline{R}\wedge\underline{k} = \underline{R}\wedge(-Mg\underline{k}) = \underline{R}\wedge\underline{F}\,,$$

which is the usual definition of the moment for a single particle. Hence uniform gravitational fields have a great deal of simplicity, because we can treat a system of particles as being equivalent to a single particle of mass M existing at the centre of mass $\underline{R}$.

Example 7.1: Moments in uniform gravity

Question: There are three particles of mass $m_1 = 2$, $m_2 = 1$, $m_3 = 2$ positioned at $\underline{r}_1 = (1, 0, 0)$, $\underline{r}_2 = (-1, 0, 0)$ and $\underline{r}_3 = (-1, 1, 0)$ respectively, in a Cartesian coordinate system relative to a fixed origin O, with unit basis vectors $\underline{i}, \underline{j}, \underline{k}$. If the particles experience a uniform gravitational force $\underline{F}_i = -m_i g \underline{j}$ for some constant g, find the total moment about the origin of the system.

Answer: There are two approaches we can take, one way is to compute the total moment from the definition:

$$\underline{L}_O = \sum_{i=1}^{3} \underline{r}_i \wedge \underline{F}_i = \cdots = (0, 0, g)\,.$$

This requires three cross products, so for more than a few particles such a method can get rather long.

However, based on the results above, we know that an alternative is to compute the centre of mass and treat the system as a single particle of mass $M = m_1 + m_2 + m_3 = 5$ placed at that point.

The centre of mass is, by definition,

$$\underline{R} = \frac{1}{M}\sum_{i=1}^{3} m_i \underline{r}_i = \frac{1}{5}(-1, 2, 0)\,,$$

so the moment about the origin must be

$$\underline{L}_O = \underline{R}\wedge\underline{F} = \underline{R}\wedge\sum_{i=1}^{3}\underline{F}_i = \frac{1}{5}(-1, 2, 0)\wedge(0, -5g, 0) = (0, 0, g)\,.$$

Example 7.2: (Non-uniform) gravitational attraction to a pair of particles

Question: Two fixed particles A and B, both of mass M, attract a particle C of mass m. The particles A and B have position vectors $\pm a\underline{j}$, whilst particle C is at $x\underline{i}$ in the usual notation for a Cartesian coordinate system; see figure 5. Find the total force exerted on C by the presence of A and B.

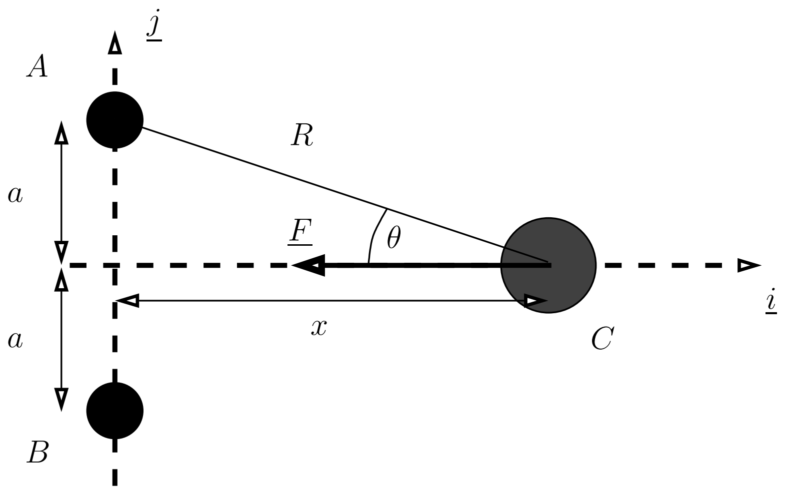


Figure 5: Attraction of a particle C to the two particles A and B.

Answer: We know from Newton's law of gravitational attraction that each of A and B attract C with a force of magnitude mMG/R^2, where

$$R = \sqrt{a^2 + x^2}\,,$$

is the separation distance and G is the gravitational constant. The vector force experienced by C is therefore

$$\begin{aligned} \underline{F} &= \frac{mMG}{R^2}\frac{\vec{CA}}{|\vec{CA}|} + \frac{mMG}{R^2}\frac{\vec{CB}}{|\vec{CB}|}\,, \\ &= \frac{mMG}{R^2}\left\{(-\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) + (-\cos\theta\,\underline{i} - \sin\theta\,\underline{j})\right\}\,, \\ &= -\frac{2mMG\cos\theta}{R^2}\,\underline{i}\,, \end{aligned}$$

where $\vec{CA}$ and $\vec{CB}$ are vectors in the directions of C to A and C to B respectively. The resultant force is parallel to the vector $\underline{i}$, which rather obviously had to be the case as the system is symmetric about the x-axis.

As $\cos\theta = x/R$ we can rewrite this result as

$$\underline{F} = -2mMG\frac{x}{(a^2 + x^2)^{3/2}}\,\underline{i}\,.$$

Note that when C is very far away from A and B, that is when $x \gg a$, we can approximate the force by the leading-order expression

$$\underline{F} \approx -\frac{2mMG}{x^2}\,\underline{i}\,.$$

So *in this limit* the system A and B acts like a single particle of mass $2M$ situated at the origin.

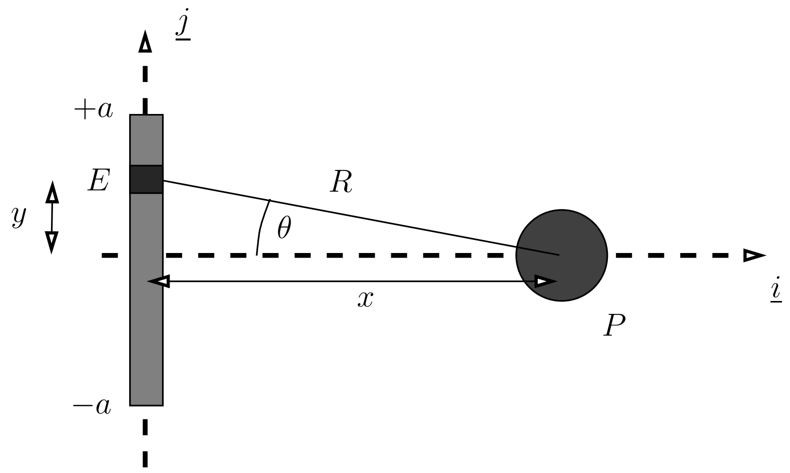


Figure 6: Attraction of a particle P to a continuous distribution of mass of length $2a$. E is a 'small' element of the rod and R is the separation distance between E and P.

Example 7.3: Attraction to a continuous distribution of mass

Question: A solid uniform rod of length $2a$ and mass M is placed along the y-axis such that its centre is at the origin of a Cartesian coordinate system; see figure 6. A particle P has a position vector $x\underline{i}$ and is of mass m. Find the gravitational attraction of P to the rod.

Answer: The total mass of the rod is M and this is distributed uniformly throughout its length. Hence, for a small element E of the rod, the mass of that single element is

$$M\left(\frac{\mathrm{d}y}{2a}\right),$$

where dy is the length of the element E. Hence the gravitational attraction exerted on the particle P by the presence of the element E is

$$\frac{m\left[M\left(\frac{dy}{2a}\right)\right]G}{R^2}\frac{\vec{PE}}{|\vec{PE}|},$$

where R is the separation distance between P and E.

To compute the *total* attractive force, we must now integrate over all such infinitesimal elements in the rod:

$$\underline{F} = -\left\{\int_{y=-a}^{y=a}\frac{mMG}{2aR^2}\cos\theta\,\mathrm{d}y\right\}\underline{i};$$

again the symmetry of the problem leads to a resultant force along the line of symmetry.

To compute this integral we must be careful because, as we move through the rod integrating over each element, the separation distance R and angle θ will vary with y; in fact

$$R = \sqrt{x^2+y^2}, \quad \cos\theta = \frac{x}{R}.$$

Hence

$$\underline{F} = -\frac{mMG\underline{i}}{2a}\int_{y=-a}^{y=+a} \frac{x}{(x^2+y^2)^{3/2}}\,\mathrm{d}y\,,$$

which is a standard integral. (I don't expect you to have it memorised, but it is easy to look it up!). This leads to

$$\underline{F} = -\frac{mMG\underline{i}}{x^2}\left(1+\frac{a^2}{x^2}\right)^{-1/2}\,.$$

Again we may note that, if $x \gg a$, then $|\underline{F}| \approx mMG/x^2$, so the rod acts like a single particle of mass M placed at the centre/origin O.

Extra

Example 7.4: Attraction to a spherical distribution of mass

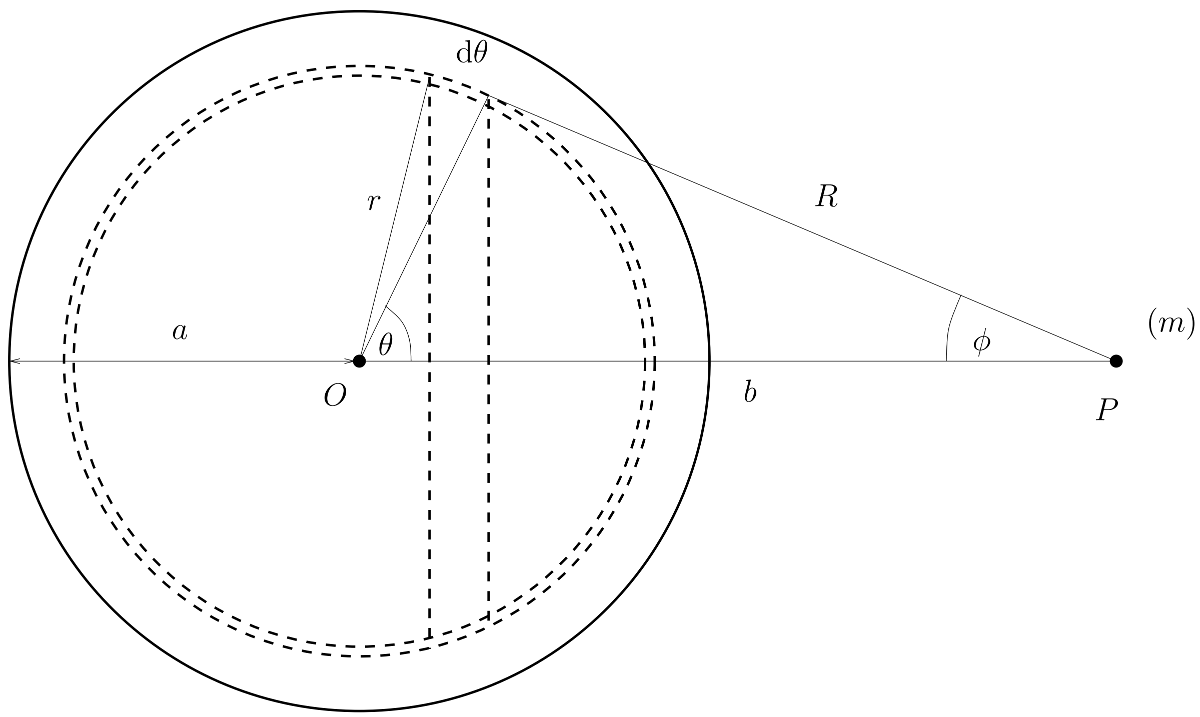


Figure 7: Attraction of a particle P of mass m to a continuous (spherical) distribution of mass of radius a and total mass M. Here we show an inner spherical shell of radius r and a slice through that shell at constant θ (shown as dotted lines); these sections are conceptually important in the integration over all parts of the sphere.

Question: Consider a spherical body of radius a, total mass M and constant density ρ; where $\rho = M/V$ with V being the volume of the sphere. Determine the force exerted on a particle P placed at a radial distance b from the centre of the sphere, where $b > a$ as shown in figure 7.

Answer: Suppose we consider an inner spherical shell of thickness $\mathrm{d}r$, then divide that shell into 'slices' (at constant θ) each of 'width' of $r\mathrm{d}\theta$; see figure 7. The mass of the slice is its volume times the density ρ:

$$M_{slice} = \rho\, 2\pi r \sin\theta\, r\mathrm{d}\theta\, \mathrm{d}r\,,$$

where $r\sin\theta$ is the radius of the slice.

By symmetry we can anticipate that the force on P is directed towards the centre of the sphere, and the magnitude of the force of attraction to the sliced section is

$$F_{slice} = \frac{mG}{R^2} M_{slice} \cos\phi,$$

where the $\cos\phi$ arises because we need the component of the force along the axis of symmetry.

The magnitude of the total attractive force is an integral over all such sliced sections. We therefore must integrate from $r = 0$ to $r = a$ (all spherical shells) and for each shell integrate from $\theta = 0$ to $\theta = \pi$ (all slices within the shell). The magnitude of the total force of attraction is therefore

$$F = \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=\pi} \frac{mG}{R^2} \cos\phi\, \rho\, 2\pi r^2 \sin\theta \mathrm{d}\theta\, \mathrm{d}r\,.$$

We can rewrite this as

$$F = 2\pi m G \int_{r=0}^{r=a} \rho r^2 \int_{\theta=0}^{\theta=\pi} \frac{\cos\phi \sin\theta}{R^2} \mathrm{d}\theta\, \mathrm{d}r\,.$$

To tackle the θ integral we note that $R^2 = (r\sin\theta)^2 + (b - r\cos\theta)^2 = r^2 - 2br\cos\theta + b^2$, and

$$\cos\phi = \frac{b - r\cos\theta}{R}\,,$$

so that

$$F = 2\pi m G \int_{r=0}^{r=a} \rho r^2 \int_{\theta=0}^{\theta=\pi} \frac{b - r\cos\theta}{R^3} \sin\theta \mathrm{d}\theta\, \mathrm{d}r\,.$$

We can change the θ integration to being over R by noting that

$$2R\frac{\mathrm{d}R}{\mathrm{d}\theta} = 2rb\sin\theta\,,$$

with limits of $R = b - r$ to $R = b + r$. Finally, on noting that,

$$b - r\cos\theta = \frac{2b}{2b}(b - r\cos\theta) = \frac{2b^2 - 2rb\cos\theta}{2b} = \frac{b^2 + (b^2 - 2rb\cos\theta)}{2b} = \frac{b^2 + R^2 - r^2}{2b}\,,$$

we can rewrite the force as

$$F = 2\pi m G \int_{r=0}^{r=a} \rho r^2 \int_{R=b-r}^{R=b+r} \frac{1}{2rb^2} \left(\frac{b^2 - r^2}{R^2} + 1 \right) \mathrm{d}R\, \mathrm{d}r\,.$$

The inner integral is now easy, leaving us with

$$F = \frac{4\pi m G}{b^2} \int_{r=0}^{r=a} \rho r^2 \mathrm{d}r = \left(\frac{4}{3}\pi a^3 \rho \right) \frac{mG}{b^2} = \frac{MmG}{b^2}\,,$$

as $4\pi a^3/3 = V$ is the volume of the sphere and $V\rho = M$ is the total mass of the sphere.

Therefore, the gravtiational attraction of a particle to a spherically distributed mass is *exactly* the same as if the entire mass of the sphere was concentrated at its centre. Unlike examples 7.2 and 7.3, there is no approximation in this case, we only require that $b > a$, **not** $b \gg a$. In fact, we could repeat the analysis above with a radially varying density $\rho = \rho(r)$ and the result would still hold.

If $b < a$ then the inner integral in the computation above becomes zero (left as an exercise!), which shows that there is no gravitational field *inside* a spherical *shell*. This is not an intuitive result, and is a consequence of the inverse-square dependence in Newton's law of gravitation. Hence, if you were to tunnel into a spherical planet, only the mass 'below' you would contribute to the gravitational force that you feel. As a direct consequence (another exercise!), this shows that the gravitational force acting on P is a maximum when $b = a$, and for $b < a$ it decreases linearly with decreasing b if the sphere has a constant density; being zero when $b = 0$.
