MATH10222, Chapter 2: Newtonian Dynamics

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These notes are provided as a revision/overview of the lectures. Any expressions/formulae that I expect you to have memorised for the examination are highlighted with a surrounding box.

1 Newton's Laws

N1. When all external influences on a particle are removed, the particle moves with constant velocity (which may be zero).

N2. When a force $\underline{F}$ acts on a particle of constant mass m, the particle moves with an acceleration $\underline{a}$ where

$$\boxed{\underline{F} = m\underline{a}\,.}$$

N3. When two particles exert forces upon eachother, these forces are (i) equal in magnitude, (ii) opposite in direction and (iii) parallel to the straight line joining the two particles.

Aside: if the mass of the particle is not constant then instead of $\underline{F} = m\underline{a}$ we have to use the alternative form $\underline{F} = \dot{\underline{p}}$ where $\underline{p}$ is the linear momentum (as defined in chapter 1). It is trivial to see that these two forms are equivalent when $\dot{m} = 0$. All the cases covered in this course have particles of constant mass.

2 Newton's 2nd law

Newton's second law for a particle P of constant mass m is

$$m\ddot{\underline{r}} = \underline{F}\,, \tag{1}$$

which is obviously a differential equation for the position vector $\underline{r}$ of the particle P. This is the 'linear momentum equation' and most of the time our goal is to solve for $\underline{r}$. If we are lucky and $\underline{F}$ is integrable, then we could just integrate twice to find $\underline{r}$, but typically this is not the case.

There are a couple of easy and useful things we can do with equation (1). We can consider:

1. the cross product of (1) with $\underline{r}$

2. the dot product of (1) with $\dot{\underline{r}}$

For the first case, the cross product with $\underline{r}$ leads to

$$\underline{r} \wedge m\ddot{\underline{r}} = \underline{r} \wedge \underline{F} = \underline{L}_O \, ,$$

by the definition of the moment $\underline{L}_O$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}(\underline{r} \wedge m\dot{\underline{r}}) = \underline{L}_O \, ,$$

which turns out to be such a useful equation that we give it a name – it's the 'angular momentum equation'.

We define the **angular momentum** of the particle to be

$$\underline{H}_O = \underline{r} \wedge m\dot{\underline{r}}$$

so that the short-hand version of the angular momentum equation is then

$$\frac{\mathrm{d}\underline{H}_O}{\mathrm{d}t} = \underline{L}_O \, .$$

We will make use of this in later parts of the course (sections 5 and 6 of this chapter and chapter 3).

The second case, the dot product with $\dot{\underline{r}}$, is discussed in the next section.

3 Work and energy

Taking the dot product of (1) with $\dot{\underline{r}}$ leads to

$$m\dot{\underline{r}} \cdot \ddot{\underline{r}} = \dot{\underline{r}} \cdot \underline{F} \, ,$$

which can be rewritten as

$$\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\underline{r}} \cdot \dot{\underline{r}}) = \dot{\underline{r}} \cdot \underline{F} \, .$$

Again this is turns out to be useful enough to give the quantities involved some names.

We define the **kinetic energy** of the particle to be

$$\boxed{T = \frac{1}{2}m|\dot{\underline{r}}|^2 \, ,}$$

so that the short-hand version of the above equation is then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \underline{F} \cdot \dot{\underline{r}} \, .$$

As a particle moves along a path between times $t = t_1$ and $t = t_2$, we can integrate with respect to time to obtain

$$\Big[T\Big]_{t_1}^{t_2} = \int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} \, \mathrm{d}t \, .$$

This is the 'work-energy equation' and relates the change in kinetic energy of the particle to the 'work done' by the applied external forces. We define the **work done**, by the force $\underline{F}$ in the time interval t_1 to t_2, as the integral

$$\int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}} \, \mathrm{d}t \, .$$

A special case

There is a special case that gets its own definition. We define a **conservative force field** $\underline{F}$ to be one for which we can write

$$\underline{F} \cdot \dot{\underline{r}} = -\frac{\mathrm{d}V}{\mathrm{d}t}\,, \tag{2}$$

for some function V. (Note that the minus sign here is simply a notational convention).

Aside: You have seen 'conservative fields' before in MATH10121 and you should review that material. These concepts are central to a number of applied mathematics courses in our programme.

In these cases, the work done integral becomes

$$\int_{t_1}^{t_2} \underline{F} \cdot \dot{\underline{r}}\,\mathrm{d}t = -\Big[V\Big]_{t_1}^{t_2},$$

and we define V to be the **potential energy** of the particle. Therefore, the work-energy equation becomes

$$\Big[T + V\Big]_{t_1} = \Big[T + V\Big]_{t_2}.$$

So the quantity $T + V$ is **conserved** (i.e., it remains unchanged) throughout the motion of the particle. Since the above statement is true for any t_1 and t_2, it is more natural to rewrite this as

$$\boxed{T + V = E\,,}$$

where E is a constant that we call the **total energy** of the particle. It is worth emphasising that $T + V = E$ applies only when the particle is acted upon by conservative forces and we can define a potential V – in example (3.2) of this section we discussed a case where this equation clearly cannot be used.

Example 3.1: Motion in a uniform gravitational field

A particle P of mass m and position vector $\underline{r}(t)$ moves in a uniform gravitational field $-g\underline{k}$, where $\{\underline{i}, \underline{j}, \underline{k}\}$ are the usual Cartesian basis vectors.

Question: By considering a general motion with $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$, show that the (total) energy of the particle is conserved.

Answer: The force in this case arises from the weight of P, $\underline{F} = -mg\underline{k}$, so the quantity $\underline{F} \cdot \dot{\underline{r}}$ becomes

$$\underline{F} \cdot \dot{\underline{r}} = -mg\underline{k} \cdot (\dot{x}\,\underline{i} + \dot{y}\,\underline{j} + \dot{z}\,\underline{k}) = -mg\dot{z}\,.$$

The work-energy equation is therefore

$$\Big[T\Big] = -mg \int \dot{z}\,\mathrm{d}t = -\Big[mgz\Big].$$

So this is one of those special cases where the work-done integral is independent of the path taken and we can define a 'potential energy' to be

$$V = mgz\,.$$

Energy is therefore conserved, and the conservation of energy equation (for motion in a uniform gravitational field) is

$$T + mgz = E\,,$$

where E is a constant.

Question: Find the position of P at time t, given that at time $t = 0$ the particle has position $\underline{r}_0$ and velocity $\underline{V}_0$.

Answer: N2 states that

$$m\ddot{\underline{r}} = \underline{F} = -mg\underline{k}\,,$$

since the only force acting on P is the weight.

Integrating the acceleration once gives the velocity

$$\dot{\underline{r}} = -g\underline{k}t + \underline{b}\,,$$

where $\underline{b}$ is a constant vector arising from the integration. Applying the initial condition for the velocity shows that $\underline{b} = \underline{V}_0$.

Integrating again to find the position leads to

$$\underline{r}(t) = -\frac{gt^2}{2}\underline{k} + \underline{V}_0 t + \underline{r}_0\,,$$

where again we have used the initial condition to determine a constant vector.

Question: Using this particular position vector, *verify* that the conservation equation is indeed satisfied. **Extra**

Answer: The conservation equation is

$$\frac{m}{2}|\dot{\underline{r}}|^2 + mgz = E\,, \quad \text{(a const)}\,,$$

where $\underline{r} = -\frac{1}{2}gt^2\underline{k} + \underline{V}_0 t + \underline{r}_0$. For the two vector constants we will use a component form via $\underline{V}_0 = V_x\underline{i} + V_y\underline{j} + V_z\underline{k}$ and $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$.

Using this position vector, the conservation equation reduces to

$$\frac{m}{2}\left[V_x^2 + V_y^2 + (V_z - gt)^2\right] + mg\left(V_z t - \frac{gt^2}{2} + z_0\right) = E\,,$$

or

$$\frac{m}{2}\left[V_x^2 + V_y^2 + V_z^2 - 2V_z gt + g^2t^2\right] + mg\left(V_z t - \frac{gt^2}{2} + z_0\right) = E\,.$$

The time dependent terms cancel to leave

$$\frac{m}{2}\left[V_x^2 + V_y^2 + V_z^2\right] + mgz_0 = E\,,$$

which is the (constant) total energy of the system $E = \frac{1}{2}m|\underline{V}_0|^2 + mgz_0$.

Example 3.2: Motion in a uniform gravitational field with air resistance

Suppose we reconsider the previous example with $\underline{r}_0 = \underline{0}$. Rather than P just experiencing a force due to (uniform) gravity, let's include an additional force that is a (very simplified!) model of 'air resistance'. To model this resistance force we will assume that it is linearly proportional to the particle's speed and acts against the direction of motion.

The force exerted upon P in this case is then

$$\underline{F} = -mg\underline{k} - m\gamma\dot{\underline{r}}\,,$$

where γ is a constant associated with the resistance force (per unit mass).

Question: Find the position of P at time t.

Answer: In this case N2 leads to

$$m\ddot{\underline{r}} = -mg\underline{k} - m\gamma\dot{\underline{r}}\,,$$

so that

$$\ddot{\underline{r}} + \gamma\dot{\underline{r}} = -g\underline{k}\,.$$

This is a (vector) second-order constant coefficient ODE, so we can solve it using any of the techniques learned in the first half of this course. One way forward is to simply integrate once and use the initial conditions to show that

$$\dot{\underline{r}} + \gamma\underline{r} = \underline{V}_0 - gt\underline{k}\,.$$

From here we could choose to use an integrating factor:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{r}e^{\gamma t}\right) = (\underline{V}_0 - gt\underline{k})e^{\gamma t}\,.$$

Integrating this (by parts) then provides

$$\underline{r}e^{\gamma t} = \frac{e^{\gamma t}}{\gamma}\underline{V}_0 - \frac{gte^{\gamma t}}{\gamma}\underline{k} + \frac{ge^{\gamma t}}{\gamma^2}\underline{k} + \underline{c}\,,$$

where $\underline{c}$ is a constant (vector) of integration.

Again we determine $\underline{c}$ from the initial conditions to find that

$$\underline{r}(t) = \frac{g}{\gamma^2}\left\{1 - \gamma t - e^{-\gamma t}\right\}\underline{k} + \frac{1}{\gamma}\left\{1 - e^{-\gamma t}\right\}\underline{V}_0\,.$$

Question: Explain the behaviour for $t \to \infty$.

Answer: It's easy to see that as $t \to \infty$, we are left with a leading-order behaviour of

$$\underline{r} \sim -\frac{gt}{\gamma}\underline{k}\,,$$

and therefore

$$\dot{\underline{r}} \sim -\frac{g}{\gamma}\underline{k}\,.$$

In other words, irrespective of the initial velocity, the ultimate behaviour of P is vertical motion at constant speed. This constant value is often called the 'terminal velocity' (it can be found directly from N2 without finding the general solution, can you see how?).

4 Motion along a line

A problem worth some discussion is one in which a particle P of mass m moves along a line and is acted upon by a force that is parallel to that line and only a function of the particle's position. In this case we can define the position vector of P as $\underline{r}(t) = x(t)\underline{i}$ where $\underline{i}$ is a constant unit vector and then the force is $\underline{F}(x) = F(x)\underline{i}$.

Newton's second law for this problem then reduces to a scalar equation (because this is a one-dimensional problem with motion confined to the line):

$$m\ddot{x} = F(x)\,. \tag{3}$$

For motion along a line it is clear that a potential energy can always be defined for any integrable scalar function $F(x)$. Using equation (2) we define

$$\begin{aligned} V &= -\int \underline{F}\cdot\dot{\underline{r}}\,\mathrm{d}t\,, \\ &= -\int F(x)\,\mathrm{dx}\,. \end{aligned}$$

Hence, so long as we can integrate the scalar force $F(x)$ we can define a potential energy, and therefore we can tackle these one-dimensional problems just using the conservation of energy equation

$$T + V = E\,,$$

where E is the constant total energy, and $T = (m\dot{x}^2)/2$ is the kinetic energy. Typically the constant E will be determined by the initial conditions for the particle (i.e., its initial position and speed).

Example 4.1: Bounded motion

A particle P of mass $m = 2$ moves along the positive x-axis under the influence of a force $F(x)\,\underline{i}$, where

$$F(x) = \left(\frac{4}{x^2} - 1\right)\,.$$

Question: Find and sketch the potential $V(x)$.

Answer: For motion restricted to a line the potential is

$$V(x) = -\int F\,\mathrm{d}x = -\int \left(\frac{4}{x^2} - 1\right)\,\mathrm{d}x = \frac{4}{x} + x + V_0,$$

where V_0 is a constant of integration. The exact value of V_0 will not affect the motion so, without loss of generality, $V_0 = 0$ and the potential is $V(x) = 4/x + x$. The sketch is shown in figure 1.

Question: Find the energy equation that describes the motion and find the minimum possible value of the energy.

Answer: The energy equation is

$$T + V = E \quad \Rightarrow \quad \frac{1}{2}m\dot{x}^2 + \frac{4}{x} + x = E.$$

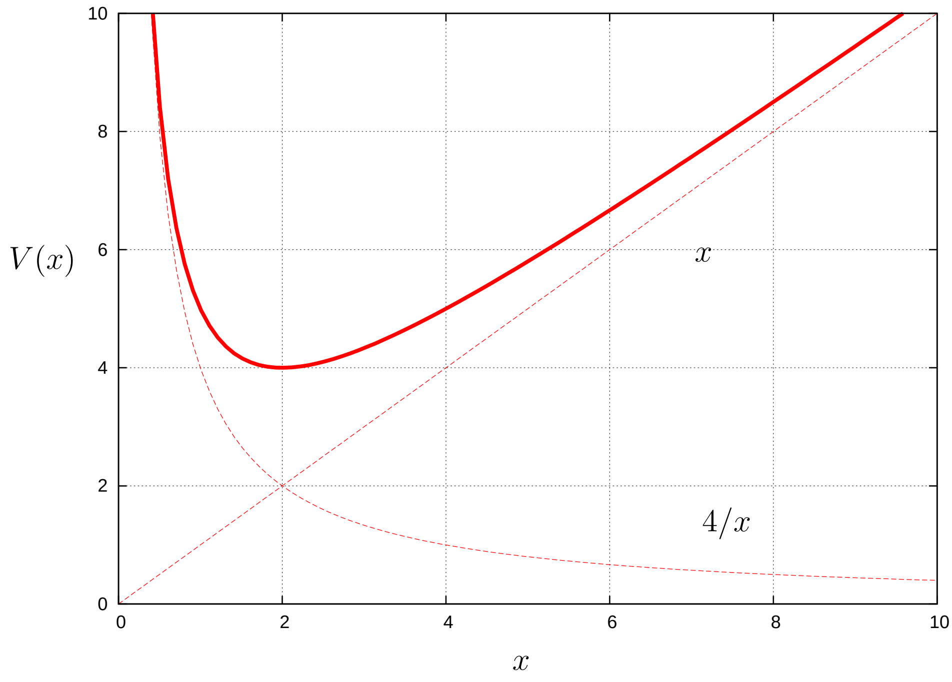

Figure 1: The potential well for example 4.1.

The mass of the particle is $m = 2$, so the energy equation becomes

$$\dot{x}^2 + \frac{4}{x} + x = E, \quad \text{a constant}$$

Aside: If we had not chosen $V_0 = 0$ then the energy equation would be

$$\dot{x}^2 + \frac{4}{x} + x = E - V_0 = \hat{E}, \quad \text{a constant,}$$

as before.

It is clear that

$$E = \dot{x}^2 + V(x) \geq V(x) \geq V_{\text{min}} .$$

Thus the total energy has a minimum at the minimum of the potential energy

$$E \geq E_{\text{min}} = V_{\text{min}} = 4.$$

The minimum energy is attained if the particle is at rest ($\dot{x} = 0$) at $x = 2$, the bottom of the potential well.

Question: If the particle is released from rest at $x = 4$, show that it will oscillate with turning points at $x = 1$ and $x = 4$.

Answer: If the particle is released from rest ($\dot{x} = 0$) at $x = 4$ the initial energy is

$$E = 0^2 + \frac{4}{4} + 4 = 5 .$$

The energy is constant for all time and so the equation governing the motion for $t \geq 0$ is then

$$\dot{x}^2 + \frac{4}{x} + x = 5.$$

The turning points in the particle motion are when $\dot{x} = 0$, so

$$0 + \frac{4}{x} + x = 5 \quad \Rightarrow \quad x^2 - 5x + 4 = 0 \quad \Rightarrow \quad (x-1)(x-4) = 0.$$

Thus, the particle is at rest at $x = 1$ and $x = 4$ and the particle oscillates between these two points.

Example 4.2: Unbounded motion

A particle P of unit mass moves along the positive x-axis under the influence of a force $F(x)\,\underline{i}$, where

$$F(x) = \left(\frac{36}{x^3} - \frac{9}{x^2}\right).$$

Question: Find and sketch the potential $V(x)$.

Answer: The potential is

$$V = -\int F \,\mathrm{d}x = -\int \left(\frac{36}{x^3} - \frac{9}{x^2}\right) \mathrm{d}x = \frac{18}{x^2} - \frac{9}{x} + V_0,$$

and again we choose $V_0 = 0$, so $V(x) = 18/x^2 - 9/x$. The sketch is shown in figure 2.

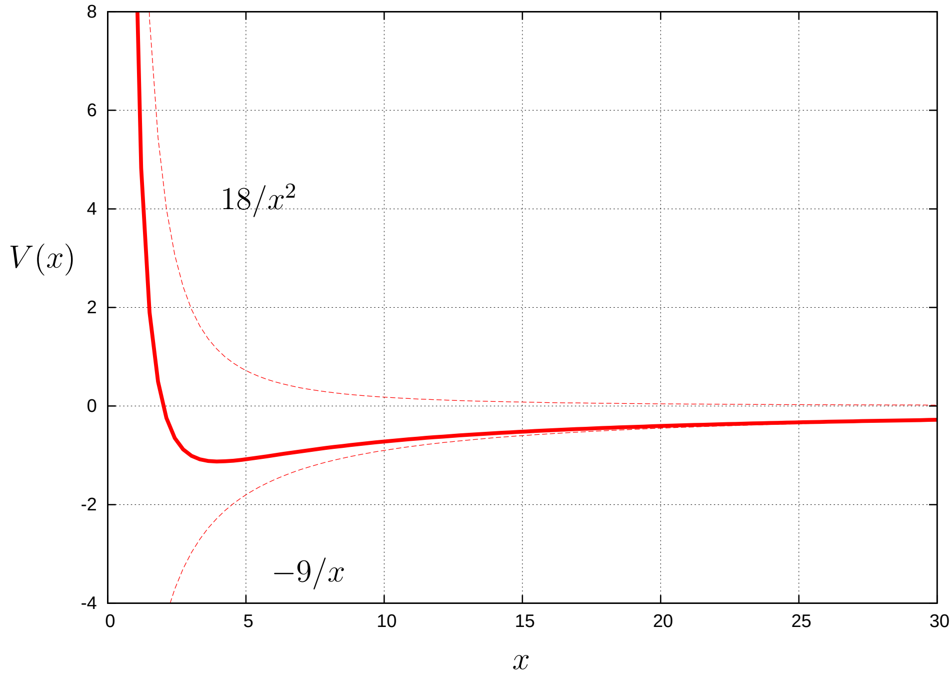

Figure 2: The potential well for example 4.2.

Question: Find the minimum energy of the system.

Answer: The energy equation is

$$\frac{1}{2} m\dot{x}^2 + V(x) = E \quad \Rightarrow \quad \frac{1}{2}\dot{x}^2 + \frac{18}{x^2} - \frac{9}{x} = E\,.$$

As before, the minimum possible energy is always the minimum possible potential energy, which occurs at $x = 4$,

$$E_{min} = V(4) = -\frac{9}{8}.$$

Question: Describe the two possible types of motion and find the energy threshold that divides them.

Answer: Oscillatory motion is only possible if there are two points at which $\dot{x} = 0$, and in between these two points we must have $E > V(x)$. This is only possible if a horizontal line at a value of E intersects $V(x)$ twice either side of a minimum.

From the sketch of the solution, this is only possible if $E < 0$. Thus, if $E < 0$, there will be oscillatory motion, but if $E > 0$, there will be unbounded motion. We say that the particle can "escape the potential well" if $E > 0$.

Question: If the particle is projected from the point $x = 4$ with initial speed $|\dot{x}| = 0.5$ show that it oscillates between the two points $x = 3$ and $x = 6$.

Answer: At the initial time $|\dot{x}| = 1/2$ and $x = 4$, so the energy is

$$E = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 + \frac{18}{4^2} - \frac{9}{4} = \frac{1}{8} - \frac{9}{8} = -1.$$

Thus, $E < 0$, and we expect the particle to oscillate between two points. The limits of particle motion occur when $\dot{x} = 0$, in which case

$$\frac{18}{x^2} - \frac{9}{x} = -1 \quad \Rightarrow \quad x^2 - 9x + 18 = 0 \quad \Rightarrow \quad (x-6)(x-3) = 0,$$

and the particle oscillates between $x = 3$ and $x = 6$.

Example 4.3: Period of oscillation

Question: For the same problem described in example 4.3 above, find the time taken for a single oscillation as a definite integral when $E = -1$. (ie., find the period of the motion).

Answer: The time taken for the particle to move between the two extremes of its travel can be found by working with the energy equation when $E = -1$.

$$\frac{1}{2}\dot{x}^2 + \frac{18}{x^2} - \frac{9}{x} = -1 \quad \Rightarrow \quad \dot{x} = \pm\sqrt{2}\left[-1 - \frac{18}{x^2} + \frac{9}{x}\right]^{1/2}$$

$$\Rightarrow \dot{x} = \pm\frac{\sqrt{2}}{x}\left[-x^2 + 9x - 18\right]^{1/2}.$$

$$\Rightarrow \dot{x} = \pm\frac{\sqrt{2}}{x}\left[(x-3)(6-x)\right]^{1/2}.$$

We define $t = 0$ to be when the particle is at $x = 3$ and suppose that the particle reaches $x = 6$ at $t = \tau$. When moving from $x = 3$ to $x = 6$, $\dot{x} > 0$ and so we take the positive root above. Separating variables and integrating between the two end points of the motion gives

$$\int_{x=3}^{x=6} \frac{x}{[(x-3)(6-x)]^{1/2}}\,\mathrm{d}x = \sqrt{2}\int_0^{\tau} \mathrm{d}t.$$

$$\Rightarrow \tau = \frac{1}{\sqrt{2}} \int_{x=3}^{x=6} \frac{x}{[(x-3)(6-x)]^{1/2}} \, \mathrm{d}x \, .$$

The time taken for one complete oscillation is therefore 2τ.

Example 4.4: Escape velocity

A particle of mass m moves along the $x > 0$ axis under the action of a force $\underline{F} = -(m\gamma/x^2)\underline{i}$, where $\gamma > 0$.

Question: If the particle is projected from $x = R > 0$ with velocity $U\underline{i}$ ($U > 0$), find the minimum value of U that allows the particle to escape to infinity.

Answer: As in the previous examples we can define the potential via

$$V(x) = -\int F(x) \, \mathrm{d}x = -\frac{m\gamma}{x} \, ,$$

on setting the constant of integration to zero.

We therefore know that

$$T + V = E \quad \Rightarrow \quad \frac{1}{2} m\dot{x}^2 - \frac{m\gamma}{x} = E \, ,$$

where the initial conditions determine the energy to be

$$E = \frac{m}{2}\left(U^2 - \frac{2\gamma}{R}\right) .$$

For unbounded motion we require $E > 0$, which occurs if and only if $U^2 > 2\gamma/R$. Therefore the 'escape velocity' (that is, the minimum such value of U) is

$$U_{min} = \sqrt{\frac{2\gamma}{R}} \, .$$

4.1 The phase plane

A method of visualising the solution in these 'motion along a line' problems is to consider a 'phase plane'. In this approach, each energy level is associated with a curve in the plane spanned by position (x) and velocity ($\dot{x}$). You will see more discussion of the phase plane in other courses when you meet plane-autonomous systems.

Examples: The phase plane diagram for examples 4.1 & 4.2

For example 4.1 above, the potential is $V(x) = 4/x + x$ with $x > 0$. All motion in this case is bounded, and the phase plane is shown in figure 3(a). The solutions in the phase plane consist of closed curves (parameterised by time and the energy level), corresponding to the bounded oscillations of the particle.

For example 4.2. the potential is $V(x) = 18/x^2 - 9/x$ with $x > 0$. In this case there are two types of motion. If $E < 0$ only bounded oscillations exist, but if $E > 0$ the particle will escape to infinity. In terms of the phase plane, bounded oscillations are closed loops (as above), but as the energy level increases, escape to infinity is possible, as shown in figure 3(b). Note that for a given value of $E > 0$, for large x the velocity is constant and proportional to $\pm\sqrt{2E/m}$; this follows from $T + V = E$ as $V(x) \to 0$ as $x \to \infty$.

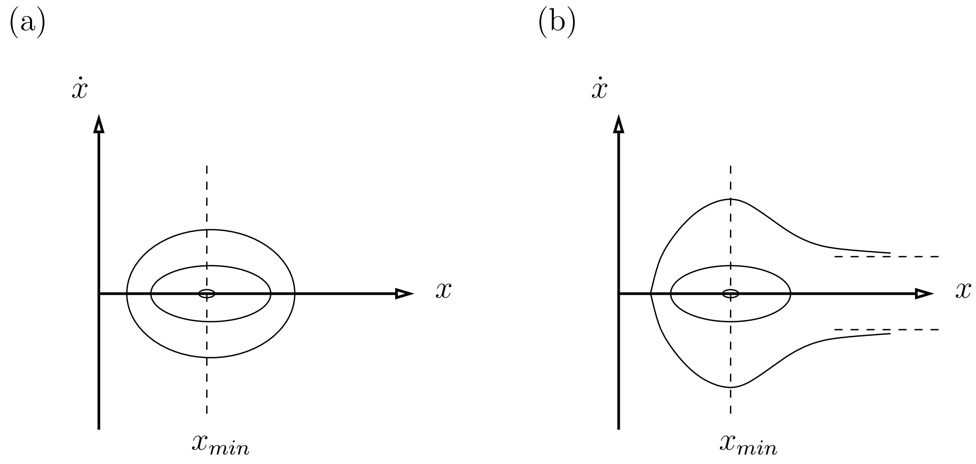

Figure 3: Phase plane diagrams for the motion types obtained in examples 4.1 (a) and 4.2 (b). In figure (b), the 'escape' orbits have a finite velocity ($\dot{x}$) as $x \to \infty$. Here $x = x_{min}$ indicates the equilibrium position where $V(x)$ takes its minimum value.

4.2 The stability of equilibrium points

Equilibrium points are defined to be those values $x = x_e$ at which the force acting on the particle is zero, that is, $F(x_e) = 0$ or equivalently, from the definition of the potential, $V'(x_e) = 0$.

The concept of 'stability' is an important one that you should know. The basic idea is that we take an equilibrium point and impose a small perturbation to the solution, so $x = x_e + \epsilon\tilde{x}(t) + \cdots$ where $\epsilon \ll 1$. We use the fact that $\epsilon \ll 1$ to ignore any terms of $O(\epsilon^2)$. We then ask the simple question: does the size of the disturbance $\tilde{x}(t)$ grow without bound as t increases? If the answer is yes, then we say that the equilibrium point x_e is **unstable**, if it is no, then we say the point is **stable**.

Aside: there are more involved definitions of 'stability', some of which you may see in later years, but we will assume this simple definition throughout this course.

Substituting this perturbation series into Newton's second law shows that

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(x_e + \epsilon\tilde{x}(t) + \cdots\right) = F\left(x_e + \epsilon\tilde{x}(t) + \cdots\right),$$

which following a Taylor series expansion, reduces to

$$m\epsilon\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}t^2} + \cdots = F(x_e) + \epsilon\tilde{x}(t)F'(x_e) + \cdots.$$

However, by definition $F(x_e) = 0$, so we are left (at leading order) with an equation governing the perturbation:

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}t^2} - \left(\frac{F'(x_e)}{m}\right)\tilde{x} = 0\,,$$

or equivalently

$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}t^2} + \left(\frac{V''(x_e)}{m}\right)\tilde{x} = 0\,,$$

As you have seen from the ODE part of this course, there are two types of solution to this system.

- If $V''(x_e) > 0$, then the characteristic equation has purely imaginary roots and the solution for $\tilde{x}(t)$ can be written in terms of the circular functions sin and cos. Hence, $\tilde{x}(t)$ does NOT grow without bound as $t \to \infty$ (because the circular functions are bounded) and the equilibrium position x_e is stable. The frequency of oscillation is $\sqrt{V''(x_e)/m}$.
- However, if $V''(x_e) < 0$ then the characteristic equation has real roots and the solution for $\tilde{x}(t)$ can be written in terms of exponential functions (or equivalently hyperbolic functions sinh and cosh). In this case $|\tilde{x}(t)|$ grows without bound as $t \to \infty$ and the equilibrium position x_e is unstable.

5 Motion confined to a plane

Having considered motion confined to a line in the previous section, we now go on to consider motion confined to a plane. At any given instant in time the particle's position relative to the origin of a coordinate system is denoted by $\underline{r}(t)$. At this same instant the particle is moving in the direction of its velocity $\dot{\underline{r}}(t)$. Hence, at time t, a plane that contains both the particle, the origin and is also tangential to the path of the particle is spanned by the vectors $\underline{r}$ and $\dot{\underline{r}}$.

- A vector normal to this plane is (by the definition of the cross product)

  $$\underline{n} = \underline{r} \wedge \dot{\underline{r}} \, .$$

  Clearly, if this normal is always pointing in the SAME direction for all values of t, then the particle is always in the same plane.

- In chapter 1, we introduced a quantity that we called the **angular momentum** $\underline{H}_O$, which was defined to be

  $$\underline{H}_O = \underline{r} \wedge m\dot{\underline{r}} \, .$$

  If angular momentum is conserved (ie. it is a constant vector), then

  $$\frac{\mathrm{d}\underline{H}_O}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\underline{r} \wedge m\dot{\underline{r}}) = \underline{0} \, ,$$

  so from the definition of $\underline{n}$ (assuming that m is constant)

  $$m\frac{\mathrm{d}\underline{n}}{\mathrm{d}t} = \underline{0} \, ,$$

  therefore $\underline{n}$ is constant and the particle must be confined to move in a (fixed) plane.

This kinematic argument leads us to the following statement:

Conservation of angular momentum $\Rightarrow$ the motion is confined to a plane.

However the implication is only one way, that is:

Motion confined to a plane $\not\Rightarrow$ conservation of angular momentum.

Aside: This subtle distinction arises because we only need $\underline{n}$ to have a constant *direction* for the motion to be in a plane, whereas conservation of angular momentum is a stronger condition that also leads to $|\underline{n}|$ being constant.

Example 5.1: Harmonic oscillator

A particle P of mass m is acted upon by a force

$$\underline{F} = -m\omega^2 \underline{r}\,,$$

for some constant ω. (This force attracts to the origin, and is linearly proportional to distance from the origin. In this case N2 leads to $\ddot{\underline{r}} + \omega^2 \underline{r} = \underline{0}$, which is the "harmonic equation".)

Question: Prove that the angular momentum is conserved in the motion of P. (ie., show that $\underline{H}_O$ is a constant vector).

Answer: Recall that the angular momentum equation is

$$\frac{\mathrm{d}\underline{H}_O}{\mathrm{d}t} = \underline{L}_O\,,$$

and

$$\underline{L}_O = \underline{r} \wedge \underline{F}\,.$$

Hence, using the force defined in the question, we find that

$$\underline{L}_O = \underline{r} \wedge \underline{F} = \underline{r} \wedge (-m\omega^2 \underline{r}) = -m\omega^2\, \underline{r} \wedge \underline{r} = \underline{0}\,,$$

as the cross product of any vector with itself is zero.

Therefore

$$\frac{\mathrm{d}\underline{H}_O}{\mathrm{d}t} = \underline{0}\,,$$

which means that $\underline{H}_O$ must be a constant vector. So angular momentum is conserved, and then P must be confined to move in a plane.

Example 5.2: Damped harmonic oscillator

Suppose we repeat the previous example, but include a resistance term (proportional to the velocity) in the definition of the force:

$$\underline{F} = -m\omega^2 \underline{r} - m\gamma\dot{\underline{r}}\,,$$

where $\gamma > 0$ is a constant.

Question: (i) Show that angular momentum is NOT conserved in this case. (ii) However, prove that the particle is still confined to move in a plane.

Answer: As before, part(i) of the question is simple:

$$\underline{L}_O = \underline{r} \wedge \underline{F} = \underline{r} \wedge (-m\omega^2 \underline{r} - m\gamma\dot{\underline{r}}) = -m\omega^2\, \underline{r} \wedge \underline{r} - m\gamma \underline{r} \wedge \dot{\underline{r}}\,,$$

in this case the first term vanishes as before, but the second term remains:

$$\underline{L}_O = -m\gamma \underline{r} \wedge \dot{\underline{r}}\,,$$

which is not zero in general.

Therefore

$$\frac{\mathrm{d}\underline{H}_O}{\mathrm{d}t} = -m\gamma \underline{r} \wedge \dot{\underline{r}}\,,$$

and $\underline{H}_O$ is not a constant vector (angular momentum is not conserved).

To answer part (ii) we use the definition of $\underline{H}_O = \underline{r} \wedge m\dot{\underline{r}}$. The particle's position and velocity must then satisfy

$$\frac{\mathrm{d}(\underline{r} \wedge m\dot{\underline{r}})}{\mathrm{d}t} = -m\gamma \underline{r} \wedge \dot{\underline{r}} \,.$$

On writing $\underline{y} = \underline{r} \wedge \dot{\underline{r}}$ (just to make the equation look simpler) this becomes

$$\dot{\underline{y}} + \gamma \underline{y} = \underline{0} \,,$$

a first-order ODE for $\underline{y}$, which is easy to solve:

$$\underline{y} = \underline{r} \wedge \dot{\underline{r}} = \underline{A} e^{-\gamma t} \,,$$

where $\underline{A}$ is a constant vector.

Therefore, although angular momentum is NOT conserved, the vector $\underline{r} \wedge \dot{\underline{r}}$ does still have a constant *direction* (ie., $\underline{A}$). Thus P must be confined to a plane whose normal is $\underline{A}$.

Aside: The point of this example is to prove that motion in a plane does not imply conservation of angular momentum.

6 Central fields of force

For a particle of mass m, a central field of force is defined to be a force of the form

$$\underline{F} = mf(r)\hat{\underline{r}} \,, \tag{4}$$

where $\underline{r} = r\hat{\underline{r}}$ in the usual notation discussed in Chapter 1, Section 5. For a force in this form, there is no moment about the origin, $\underline{L}_O = \underline{0}$. Therefore angular momentum must be conserved and the motion must also be confined to a plane.

Using the definition of $\underline{r}$ and $\dot{\underline{r}}$ from Chapter 1, the 'angular momentum' is (from its definition)

$$\underline{H}_O = \underline{r} \wedge \dot{\underline{r}} = r\hat{\underline{r}} \wedge (\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}) = r^2\dot{\theta}\underline{k} \,,$$

where $\underline{k} = \hat{\underline{r}} \wedge \hat{\underline{\theta}}$ is a unit vector perpendicular to the plane of motion. As the angular momentum is a conserved, we can rewrite this as the single scalar equation

$$r^2\dot{\theta} = H_O \,,$$

where H_O is a constant for all time (set by the initial conditions).

Similarly, for a central field of force we can always define a potential energy function

$$V(r) = -\int f(r)\,\mathrm{d}r \,,$$

which follows from the same argument applied to one-dimensional motion in Section 4 above.

We therefore obtain a conservation of energy (per unit mass) equation for motion in a plane:

$$\frac{1}{2}|\dot{\underline{r}}|^2 + V(r) = E \quad \rightarrow \quad \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \,, \tag{5}$$

where E is a constant determined by the initial conditions.

An alternative (notational) approach is to define a quantity $\bar{V}$, which is the **effective potential**

$$\bar{V}(r) = V(r) + \frac{H_O^2}{2r^2},$$

Aside: Note that these expressions are all independent of the mass of the particle m, this is because we included a factor m in the definition of the force field (4). This is simply a notational convenience inspired by the application of this method to gravitational fields, for which the force is indeed linearly proportional to the mass m.

Given any form of the force $f(r)$ and a value for the constant H_O, the equation $\dot{r}^2/2 + \bar{V} = E$ is an equation for the evolution of r as a function of time. These cases are then much like the problems encountered in section 4 but with x replaced by r and $V(x)$ replaced by $\bar{V}(r)$. The only extra complexity here is we (typically) have to use initial conditions to evaluate the TWO constants H_O and E.

Example 6.1: The harmonic oscillator (again)

A particle P moves in a central field of force with

$$f(r) = -\omega^2 r,$$

in the usual notation, where ω is a constant.

Initially P is a distance d from O, and is projected with speed U at an angle α to the line OP.

Question: Find the maximum/minimum distance $|OP|$ during the motion.

Answer: This is a central field of force, so we know that the motion is confined to a plane. We also know that a potential function exists and that we have conservation of energy:

$$\frac{1}{2}\dot{r}^2 + \bar{V}(r) = E.$$

Here E is a constant and

$$\bar{V}(r) = -\int f(r)\,\mathrm{d}r + \frac{H_O}{2r^2},$$

where

$$H_O = r^2\dot{\theta},$$

is also a constant.

To make use of the conservation of energy equation we need to integrate $f(r)$ to find $\bar{V}(r)$, then we also need to find the two constants E and H_O.

The integration is trivial:

$$\bar{V}(r) = \frac{\omega^2 r^2}{2} + \frac{H_O^2}{2r^2},$$

(after setting the constant of integration to zero).

Now we determine the two constants E and H_O by considering the initial conditions. At $t = 0$ we know that $r(t = 0) = d$ together with (see Chapter 1, Section 5)

$$\dot{\underline{r}}(t = 0) = \dot{r}(t = 0)\hat{\underline{r}} + r(t = 0)\dot{\theta}(t = 0)\hat{\underline{\theta}} = U\cos\alpha\,\hat{\underline{r}} + U\sin\alpha\,\hat{\underline{\theta}}.$$

Comparing coefficients of $\underline{\hat{r}}$ and $\underline{\hat{\theta}}$ yields

$$\dot{r}(t=0) = U\cos\alpha\,, \quad d\dot{\theta}(t=0) = U\sin\alpha\,,$$

(after using $r(t=0) = d$) and therefore the angular momentum constant must be

$$H_O = dU\sin\alpha\,.$$

To find E we evaluate the energy equation at $t = 0$,

$$E = \frac{1}{2}(U^2 + \omega^2 d^2)\,.$$

So, using these definitions of $\bar{V}(r)$, E and H_O, we find that for $t \geq 0$ the particle must satisfy:

$$\frac{1}{2}\dot{r}^2 + \left(\frac{\omega^2 r^2}{2} + \frac{U^2 d^2 \sin^2\alpha}{2r^2}\right) = \frac{1}{2}\left(\omega^2 d^2 + U^2\right)\,.$$

The distance $|OP| = r$, so a maximum/minimum value $r = r_m$ exists when $\dot{r} = 0$, which leads to:

$$\omega^2 r_m^2 + \frac{U^2 d^2 \sin^2\alpha}{r_m^2} = \omega^2 d^2 + U^2\,.$$

After multiplying throughout by r_m^2, this results in a quadratic for r_m^2 with solutions

$$r_m^2 = \frac{1}{2\omega^2}\left((\omega^2 d^2 + U^2) \pm \sqrt{\left\{(\omega^2 d^2 + U^2)^2 - 4\omega^2 U^2 d^2 \sin^2\alpha\right\}}\right)\,,$$

where the plus sign is the maximum distance and the minus sign is the minimum distance.

Extra exercise: Is the term $\{.\} \geq 0$? Is the right hand side ≥ 0?

7 The path equation for central fields of force

Solving for $r(t)$ when the motion is confined to a plane is not conceptually difficult, but can sometimes get a little involved. Often we are more interested in the path of the particle $r(\theta)$ rather than the parameterisation of the two coordinates with time, i.e., $r(t)$ and $\theta(t)$.

To derive an equation for the path of the particle we transform the variables from $(r,t) \to (u,\theta)$, where $u = 1/r$. Hence,

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}(u^{-1})}{\mathrm{d}t} = \frac{\mathrm{d}(u^{-1})}{\mathrm{d}\theta}\dot{\theta} = -u^{-2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta} = -H_O\frac{\mathrm{d}u}{\mathrm{d}\theta}\,,$$

after substituting for the constant angular momentum via

$$H_O = r^2\dot{\theta} = \dot{\theta}u^{-2}\,.$$

We can repeat this process to find $\ddot{r}$

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-H_O\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-H_O\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\dot{\theta} = -H_O^2 u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\,.$$

Substituting into the radial component of N2:

$$\ddot{r} - r\dot{\theta}^2 = f(r)\,,$$

then gives the **path equation**:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{f(\frac{1}{u})}{H_O^2 u^2}\,. \tag{6}$$

Here H_O is the constant magnitude of the angular momentum and f is the functional form of the central field of force, as given in equation (4).

7.1 Using the initial conditions

Typically we know r and $\dot{r}$ at $t = 0$, but our equation is for $u(\theta)$, so we have to do a little work to convert typical dynamical initial conditions before they can be used to solve the path equation.

- We can arbitrarily choose any line through the origin to correspond to $\theta = 0$, so it makes sense to choose the particle to be at $\theta = 0$ at $t = 0$. Hence any conditions at $t = 0$ become conditions at $\theta = 0$.
- If we know $r(t = 0)$ then we know $u(\theta = 0) = 1/r(t = 0)$.
- Using

  $$\dot{r} = -H_O \frac{\mathrm{d}u}{\mathrm{d}\theta}\,,$$

  (see above) we can evaluate it at $t = 0$ (ie., $\theta = 0$) to show that

  $$\left.\frac{\mathrm{d}u}{\mathrm{d}\theta}\right|_{\theta=0} = -\frac{\dot{r}(t=0)}{H_O}\,. \tag{7}$$

You are not expected to memorise either the path equation or the expression above for the condition on $u'(\theta = 0)$. As I said at many points in the course, this course is not about developing your ability to memorise equations, it is about you understanding how these concepts all fit together.

Example 7.1: An inverse-cube attracting field (part 1)

Consider a particle P moving in a central field of force with

$$f(r) = -\frac{\gamma}{r^3}\,,$$

where $\gamma > 0$ is a constant. Initially P is a distance d from O and is projected with a velocity

$$\underline{V}_0 = \frac{\sqrt{\gamma}}{d}\,\underline{\hat{\theta}}\,,$$

Question: Find the path taken by P.

Answer: Clearly we have to solve (6), hence we need to start by working out the constant H_O. As H_O is a constant we can use the information at $t = 0$ to evaluate it:

$$H_O = r^2\dot{\theta} = d^2\dot{\theta}(t=0)\,,$$

but we still need to find $\dot{\theta}(t = 0)$.

We know that the initial velocity is

$$\underline{\dot{r}}(t=0) = \dot{r}(t=0)\underline{\hat{r}} + r(t=0)\dot{\theta}(t=0)\underline{\hat{\theta}} = \underline{V}_0 = \frac{\sqrt{\gamma}}{d}\,\underline{\hat{\theta}}\,,$$

using the question and the definition of $\underline{\dot{r}}$ in plane polars. So comparing coefficients of $\underline{\hat{r}}$ and $\underline{\hat{\theta}}$ gives:

$$\dot{r}(t=0) = 0\,, \quad d\,\dot{\theta}(t=0) = \frac{\sqrt{\gamma}}{d}\,,$$

since $r(t = 0) = d$.

Hence

$$H_O = \sqrt{\gamma}\,,$$

and the path equation in this case becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = 0\,,$$

with a general solution

$$u = A\theta + B\,.$$

To evaluate A and B we:

- Choose $\theta = 0$ to be the line OP when $t = 0$.
- Using $r(t = 0) = d$ we then find $u(\theta = 0) = d^{-1}$.
- Using $\dot{r}(t = 0) = 0$ and (7) we then find $u'(\theta = 0) = 0$.

Clearly therefore $A = 0$ and $B = 1/d$, so $u = 1/d$ or equivalently $r = d$. So the path of P is a circle of radius d.

7.2 Example: An inverse-cube attracting field (part 2)

Suppose we re-do the previous example, but we change the initial velocity of P to be

$$\underline{V}_0 = -\frac{\sqrt{\gamma}}{d}\,\underline{\hat{r}} + \frac{\sqrt{\gamma}}{d}\,\underline{\hat{\theta}}\,.$$

Proceeding through the analysis as in the previous example we find that the only change is that now

$$\dot{r}(t=0) = -\frac{\sqrt{\gamma}}{d}\,,$$

whereas before this was zero. The angular momentum constant is the same ($H_O = \sqrt{\gamma}$) and therefore the path equation is also unchanged, with the same general solution

$$u = A\theta + B\,.$$

Now because $\dot{r}(t=0) \neq 0$ we have changed one of the initial conditions to

$$u'(\theta = 0) = -\frac{\dot{r}(t=0)}{H_O} = \frac{1}{d},$$

(rather than $u'(\theta = 0) = 0$ as in the previous example). Therefore $A = 1/d$ and $B = 1/d$ leading to

$$u = \frac{\theta + 1}{d},$$

or equivalently

$$r = \frac{d}{\theta + 1}.$$

Now rather than a closed orbit, this path is a spiral which tends ultimately towards the origin, with $r \to 0$ as $\theta \to \infty$.

8 Motion not confined to a plane

So far, we have discussed the special cases of motion confined to a line, then extended this to motion confined to a plane, but what about other cases?

When the moment is zero

$$\underline{L}_O = \underline{0},$$

the angular momentum equation tells us that

$$\frac{\mathrm{d}\underline{H}_O}{\mathrm{d}t} = \underline{L}_O = \underline{0},$$

and therefore that $\underline{H}_O$ is a **constant** vector. In this case we get a great deal of simplification, in particular we know that the motion of a particle is then confined to a plane.

Sometimes $\underline{L}_O \neq \underline{0}$, but a **component** of the moment may vanish (instead of all three components). In these cases we can similarly see that a component of $\underline{H}_O$ will be constant (again, instead of all three components). In some cases this single constant component can help us in solving problems.

8.1 Example: The spherical pendulum

A particle P of mass m is attached to an origin O by a (tight) light inextensible string of length a (ie. the string does not stretch nor does it have mass). P moves in a uniform gravitational field of strength g.

Question: Show that the $\underline{k}$ component of $\underline{H}_O$ is conserved, where $\underline{k}$ is a unit vector pointing vertically upwards.

Answer: The forces acting on P are the tension in the string and the weight. So

$$\underline{F} = \underline{T} - mg\underline{k}.$$

Clearly P is not confined to move in a plane, it is rather confined to move on a sphere of radius a (assuming the string is never 'slack'). So we will describe the position of P with a vector $\underline{q}$:

$$\underline{q} = r\underline{\hat{r}} + z\underline{k},$$

where $\underline{\hat{r}}$ is the usual radial unit vector and z represents the out of plane position.[1]

The moment is defined in the usual way

$$\underline{L}_O = \underline{q} \wedge \underline{F} = \underline{q} \wedge (\underline{T} - mg\underline{k}) = \underline{q} \wedge \underline{T} - mg\underline{q} \wedge \underline{k}\,.$$

So the $\underline{k}$ component of $\underline{L}_O$ is

$$\underline{L}_O \cdot \underline{k} = (\underline{q} \wedge \underline{T}) \cdot \underline{k} - mg(\underline{q} \wedge \underline{k}) \cdot \underline{k}\,.$$

Clearly $\underline{q} \wedge \underline{T} = \underline{0}$ as both vectors are colinear, and $(\underline{q} \wedge \underline{k}) \cdot \underline{k} = \underline{0}$ because $(\underline{q} \wedge \underline{k})$ is perpendicular to $\underline{k}$ by the definition of the cross product. Hence $\underline{L}_O \cdot \underline{k} = 0$ and therefore the angular momentum equation, after a dot-product with $\underline{k}$ gives us

$$\frac{\mathrm{d}(\underline{H}_O \cdot \underline{k})}{\mathrm{d}t} = \underline{L}_O \cdot \underline{k} = 0\,,$$

so the $\underline{k}$ component of $\underline{H}_O$ is a constant.

Other examples: See example sheet 4 for additional examples involving a particle rolling on the inside of a surface of revolution. The “hard(er)” question on that examples sheet will lead you through exactly how we can make use of the result above in solving for the motion of a particle.

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[1] I use the notation $\underline{q}$ (rather than the $\underline{r}$ used previously) in order to emphasise that this is *not* a particle confined to a plane.