MATH10222, Chapter 4: Frames of Reference

Dr. R.E. Hewitt, http://hewitt.ddns.net
Last modified: 2017-05-02

These notes are provided as a revision/overview of the lectures. Any expressions/formulae that I expect you to have memorised for the examination are highlighted with a surrounding box.

Definition: An "inertial frame of reference" is a coordinate system in which Newton's laws hold.

1 Motion relative to a translating origin

Suppose that we have an inertial frame of reference centred about an origin O. By the definition of the inertial frame, we know that a particle P of constant mass m and position vector $\underline{r}(t)$ (relative to O) satisfies Newton's second law:

$$m\ddot{\underline{r}} = \underline{F}\,,$$

where $\underline{F}$ is the resultant force acting upon P.

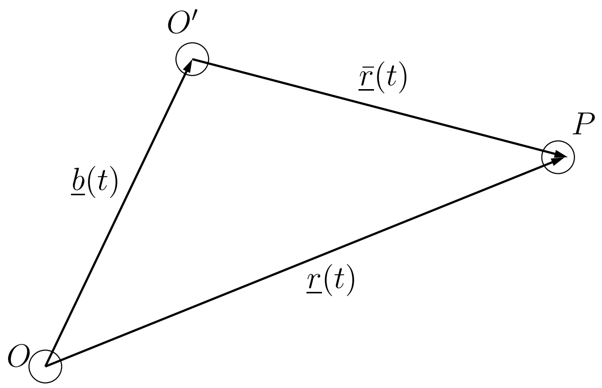

Figure 1: O is an origin in an inertial frame of reference, whereas O' is moving relative to O.

Now suppose there is another coordinate system that is defined relative to a *moving* origin O' as in figure 1. Let's further suppose that the position of O' relative to O is $\underline{b}(t)$ and that the position vector of P relative to O' is $\bar{\underline{r}}(t)$. We therefore have the relationship

$$\underline{r}(t) = \underline{b}(t) + \bar{\underline{r}}(t)\,,$$

and therefore Newton's second law reduces to

$$m\frac{\mathrm{d}^2\bar{\underline{r}}}{\mathrm{d}t^2} = \underline{F} - m\ddot{\underline{b}}\,.$$

Thus, relative to the *moving* frame of reference centred at O', the particle feels an effective force of

$$\underline{F} - m\ddot{\underline{b}}\,.$$

Newton's second law clearly only holds in the moving coordinate system if

$$\ddot{\underline{b}} = \underline{0}\,,$$

that is, if O' is moving at constant velocity relative to O. We refer to the moving coordinate system as a "non-inertial frame of reference" whenever $\ddot{\underline{b}} \neq \underline{0}$.

1.1 Example: 'zero-g' motion in a non-inertial frame of reference

Quite often a particle will be referred to as being 'weightless' when in fact it is still being acted upon by a gravitational acceleration. This is commonly the case in non-inertial frames of reference where the 'observer' and the particle are both in free fall.

For example, consider a particle P of mass m in a uniform gravitational field. The force acting on P comes from its weight $\underline{F} = -mg\underline{k}$ in the obvious notation. Clearly if our frame of reference (i.e., O') is also moving in the $\underline{k}$ direction with the same acceleration $-g$ (m/s/s), then we have

$$\ddot{\underline{b}} = -g\underline{k}\,.$$

As above let's denote the position vector of P relative to O' as $\underline{\bar{r}}$. Thus, in the non-inertial frame of reference the appropriate equation of motion is

$$m\frac{\mathrm{d}^2\underline{\bar{r}}}{\mathrm{d}t^2} = \underline{F} - m\ddot{\underline{b}} = -mg\underline{k} - m(-g\underline{k}) = \underline{0}\,,$$

and we may view the particle in this frame as (effectively) being free from any external force.

2 Two-dimensional rotating frames of reference

Instead of a coordinate system that is translating, we now consider the more complicated case of a rotating coordinate system. Consider two coordinate systems as shown in figure 2:

- A Cartesian coordinate system $\{\underline{i}, \underline{j}, \underline{k}\}$ centred at an origin O in an inertial frame of reference.
- A second Cartesian coordinate system $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ that is also centred at the origin O, but which is rotating relative to $\{\underline{i}, \underline{j}, \underline{k}\}$.

If we assume that the coordinate system $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is rotating about the $\underline{k}$ axis, with $\underline{e}_3 = \underline{k}$ and a rotation angle of $\theta(t)$, then we have (from figure 2):

$$\begin{aligned}
\underline{e}_1 &= \cos\theta\underline{i} + \sin\theta\underline{j}\,,\\
\underline{e}_2 &= -\sin\theta\underline{i} + \cos\theta\underline{j}\,,\\
\underline{e}_3 &= \underline{k}\,.
\end{aligned}$$

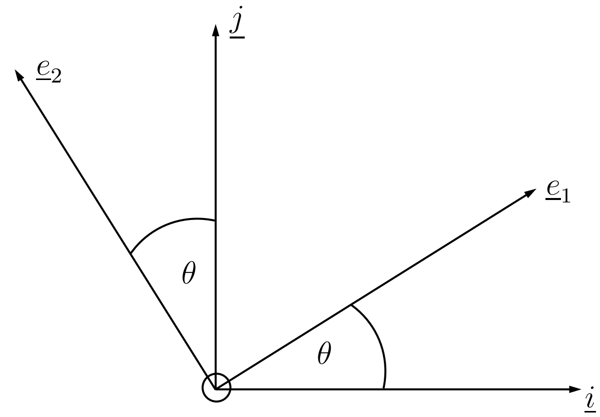

Figure 2: A Cartesian coordinate system $\underline{i}, \underline{j}, \underline{k}$ ($\underline{k}$ out of the plane) relative to an origin O, together with a second coordinate system $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ that is rotated by an angle $\theta(t)$ about the axis $\underline{e}_3 = \underline{k}$.

We're interested in the rate of change of the basis vectors $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$, which is easy to determine via

$$\begin{aligned} \dot{\underline{e}}_1 &= \frac{\mathrm{d}\underline{e}_1}{\mathrm{d}t} = \frac{\mathrm{d}\underline{e}_1}{\mathrm{d}\theta}\dot{\theta} = \dot{\theta}(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})\,, \\ \dot{\underline{e}}_2 &= \frac{\mathrm{d}\underline{e}_2}{\mathrm{d}t} = \frac{\mathrm{d}\underline{e}_2}{\mathrm{d}\theta}\dot{\theta} = \dot{\theta}(-\cos\theta\,\underline{i} - \sin\theta\,\underline{j})\,, \\ \dot{\underline{e}}_3 &= \underline{0}\,. \end{aligned}$$

We note that the above expressions are all equivalent to

$$\dot{\underline{e}}_1 = \dot{\theta}\underline{k} \wedge \underline{e}_1\,, \tag{1}$$

$$\dot{\underline{e}}_2 = \dot{\theta}\underline{k} \wedge \underline{e}_2\,, \tag{2}$$

$$\dot{\underline{e}}_3 = \dot{\theta}\underline{k} \wedge \underline{e}_3 = \underline{0}\,. \tag{3}$$

This is in fact a special case of a general result that we state next.

3 The angular frequency vector

The results of (1)–(3) generalise to (we do not prove it here)

$$\dot{\underline{e}}_i = \underline{\omega} \wedge \underline{e}_i\,, \tag{4}$$

for $i = 1, 2, 3$, where $\underline{\omega}$ is the 'angular frequency vector'. The magnitude $\omega = |\underline{\omega}|$ is then the rotation rate (or just 'angular frequency') of the rotating coordinate system, whist $\underline{\omega}/\omega$ is a unit vector that defines the axis of rotation. In the simpler case of (1)–(3) we simply had $\underline{\omega} = \dot{\theta}\underline{k}$ because the rotation rate was $\dot{\theta}$ and the axis was $\underline{k}$.

4 Velocity relative to a rotating frame

Suppose that we have an inertial frame of reference (labelled S). Further we suppose that we wish to use an alternative frame of reference S' that rotates relative to S with an angular frequency vector of $\underline{\omega}$.

Relative to the rotating frame S', we know the position of the particle:

$$\underline{r} = \sum_{i=1}^{3} x_i \underline{e}_i \,,$$

that is, in terms of three coordinates $x_{1,2,3}$ in the directions of the three basis vectors $\underline{e}_{1,2,3}$.

The velocity relative to the inertial frame of reference is then the rate of change of the position vector, so

$$\left.\dot{\underline{r}}\right|_S = \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{3} x_i \underline{e}_i\right) .$$

However, we have to be careful (as in Chapter 1, section 5), because in the frame S the three basis vectors $\underline{e}_{1,2,3}$ change with time as the coordinate system rotates. Therefore

$$\left.\dot{\underline{r}}\right|_S = \sum_{i=1}^{3} \left(\dot{x}_i \underline{e}_i + x_i \dot{\underline{e}}_i\right) \,,$$

but using (4) we can write this as

$$\begin{aligned} \left.\dot{\underline{r}}\right|_S &= \sum_{i=1}^{3} \left(\dot{x}_i \underline{e}_i + x_i \underline{\omega} \wedge \underline{e}_i\right) \,, \\ &= \left.\frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\right|_{S'} + \underline{\omega} \wedge \sum_{i=1}^{3} x_i \underline{e}_i \,, \\ &= \left.\dot{\underline{r}}\right|_{S'} + \underline{\omega} \wedge \underline{r} \,. \end{aligned}$$

The simply says that the velocity relative to the frame S is equal to the velocity relative to the rotating frame S' plus an extra contribution due to the rotation of S' relative to S.

5 A particle in a rotating frame of reference

As in the preceding section we suppose that S' is rotating relative to S with (*constant*) angular frequency vector $\underline{\omega}$, where S is an inertial frame. We know that (by definition of an inertial frame) the equation of motion for a particle P of mass m in the frame S is

$$m \left.\frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2}\right|_S = \underline{F} \,. \tag{5}$$

Suppose we prefer to describe the problem relative to the rotating frame S', then what is the equation of motion of P? To determine the correct equation we need the acceleration in the rotating frame.

We know from the previous section that the velocity is related by

$$\left.\frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\right|_S = \left.\frac{\mathrm{d}\underline{r}}{\mathrm{d}t}\right|_{S'} + \underline{\omega} \wedge \underline{r} \,.$$

However we need the acceleration, so differentiating this again (as before) we find that

$$\left.\frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2}\right|_{S} = \left.\frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2}\right|_{S'} + 2\underline{\omega}\wedge\left.\dot{\underline{r}}\right|_{S'} + \underline{\omega}\wedge(\underline{\omega}\wedge\underline{r})\,.$$

Substituting this into (5) we obtain

$$m\left.\frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2}\right|_{S'} = \underline{F} - 2m\underline{\omega}\wedge\left.\dot{\underline{r}}\right|_{S'} - m\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r})\,,$$

where the additional acceleration terms that arise from the rotation of S' are moved to the right-hand side of the equation.

So in a rotating frame S' we simply apply Newton's second law as usual, but include two additional "fictitious forces". We give these 'fictitious forces' some names:

- $-2m\underline{\omega}\wedge\left.\dot{\underline{r}}\right|_{S'}$ is the "Coriolis force",
- $-m\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r})$ is the "centrifugal force",

but they are purely a consequence of the rotating frame of reference.

5.1 Example: Plane polars and a rotating frame

Suppose we consider a particle P of mass m that moves in a rotating frame of reference. The angular frequency vector of the rotating frame is $\underline{\omega} = \omega\underline{k}$, for some constant ω.

You are given that, in the rotating frame of reference P moves in a plane with position vector $\underline{r}$ relative to an origin O on the axis of rotation where

$$\underline{r} = r\hat{\underline{r}}\,,$$

and $\hat{\underline{r}}$ is the usual unit vector that points radially outwards from the axis of rotation.

(Aside: from here we will drop the cumbersome notation of O', $\dot{\underline{r}}|_{S'}$ and just revert to our previous notation, recognising that this is a non-inertial frame that leads to additional (fictitious) forces.)

Question: Simplify the RHS of the vector equation of motion (in the non-inertial frame)

$$m\ddot{\underline{r}} = \underline{F} - 2m\underline{\omega}\wedge\dot{\underline{r}} - m\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r})\,,$$

Answer: To simplify things we need to determine the components of the Coriolis and centrifugal forces. This is easy to do as we are told that $\underline{\omega} = \omega\underline{k}$, $\underline{r} = r\hat{\underline{r}}$ and we know from Chapter 1, section 5 that

$$\dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}\,.$$

The Coriolis force is then

$$-2m\underline{\omega}\wedge\dot{\underline{r}} = -2m\omega\underline{k}\wedge(\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}) = -2m\omega\dot{r}\hat{\underline{\theta}} + 2m\omega r\dot{\theta}\hat{\underline{r}}\,,$$

whilst the centrifugal force is

$$-m\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r}) = -mr\omega^2\underline{k}\wedge(\underline{k}\wedge\hat{\underline{r}}) = -mr\omega^2\underline{k}\wedge\hat{\underline{\theta}} = mr\omega^2\hat{\underline{r}}\,.$$

The RHS of the equation of motion is therefore

$$\underline{F} - 2m\omega\dot{r}\underline{\hat{\theta}} + 2m\omega r\dot{\theta}\underline{\hat{r}} + mr\omega^2\underline{\hat{r}}\,.$$

Question: Give two scalar equations for the polar coordinates r and θ in the case $\underline{F} = \underline{0}$. **Extra**

Answer: Using Chapter 1, section 5 for the acceleration in polar basis vectors, leads to an equation of motion in the form

$$m\left((\ddot{r} - r\dot{\theta}^2)\underline{\hat{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{\hat{\theta}}\right) = -2m\dot{r}\omega\underline{\hat{\theta}} + 2m\omega r\dot{\theta}\underline{\hat{r}} + mr\omega^2\underline{\hat{r}},\,,$$

after setting $\underline{F} = \underline{0}$ (as there is no force acting).

The vector equation therefore simplifies to two scalar equations:

$$\ddot{r} - r\dot{\theta}^2 = 2\omega r\dot{\theta} + r\omega^2\,,$$

or equivalently

$$\ddot{r} = r\omega^2\left(1 + \frac{\dot{\theta}}{\omega}\right)^2\,, \tag{6a}$$

and

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -2\dot{r}\omega\,. \tag{6b}$$

Note: It is possible to simplify these further (left as an exercise).