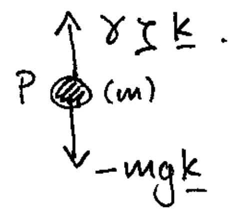
Solutions 5

Q1 (i) If $\underline{R} = \underline{0}$, then $O = O'$ and we are in an inertial frame with $\underline{r}(t) = z(t)\underline{k} = -(\ell + \zeta(t))\underline{k}$.

So $m\ddot{\underline{r}} = \underline{F}$



where $\underline{F} = -mg\underline{k} + \gamma\zeta(t)\underline{k}$

$$\Rightarrow \quad m\ddot{\underline{r}} = m\ddot{z}\underline{k} = -m\frac{d^2}{dt^2}(\ell + \zeta)\underline{k} = -m\ddot{\zeta}\underline{k} = -mg\underline{k} + \gamma\zeta\underline{k}$$

ie. motion is along a line, and we have the scalar eq$^{\underline{n}}$

$$\ddot{\zeta} = g - \frac{\gamma}{m}\zeta \quad \Rightarrow \quad \ddot{\zeta} + \frac{\gamma}{m}\zeta = g.$$

(ii) 2^{nd} order, linear, inhom. constant coefft. ODE.

$$\zeta = \zeta_{CF} + \zeta_{PI}, \quad \text{where } \zeta_{PI} = \frac{gm}{\gamma}$$

and $\ddot{\zeta}_{CF} + \frac{\gamma}{m}\zeta_{CF} = 0 \quad \Rightarrow \zeta_{CF} = A\cos\left(\sqrt{\frac{\gamma}{m}}\,t\right) + B\sin\left(\sqrt{\frac{\gamma}{m}}\,t\right)$

$$\Rightarrow \zeta(t) = A\cos\left(\sqrt{\frac{\gamma}{m}}\,t\right) + B\sin\left(\sqrt{\frac{\gamma}{m}}\,t\right) + \frac{mg}{\gamma}$$

An equilibrium solution is $A = B = 0 \Rightarrow \zeta = \frac{mg}{\gamma}$.

(This follows directly from simply setting $\underline{F} = \underline{0}$ too).

(iii). Now suppose that $\underline{R} = Ut\underline{k}$, then relative to O

$$\underline{r} = \underline{R}(t) + z(t)\underline{k}$$

So now $m\ddot{\underline{r}} = \gamma \zeta(t)\underline{k} - mg\underline{k}$ (as before) $\Rightarrow$

$$m(\ddot{\underline{R}} + \ddot{z}\underline{k}) = \gamma\zeta\underline{k} - mg\underline{k}.$$

but note that $\underline{R} = ut\underline{k} \Rightarrow \dot{\underline{R}} = u\underline{k}$, $\ddot{\underline{R}} = \underline{0}$ and

we obtain $$-m\ddot{\zeta} = \gamma\zeta - mg$$ once again.

$$\text{ie. } \ddot{\zeta} + \frac{\gamma}{m}\zeta = g$$

As we saw in Chapter 4, §1, uniform translation provides an inertial frame.

(iv). If $\underline{R} = \cos(\omega t)\underline{k}$, then relative to O

$$\underline{r} = \cos(\omega t)\underline{k} + z(t)\underline{k}$$

$$\Rightarrow m\frac{d^2}{dt^2}\left(\cos\omega t + z(t)\right)\underline{k} = (\gamma\zeta - mg)\underline{k}$$

$$-\omega^2 m\cos\omega t + m\ddot{z} = \gamma\zeta - mg$$

where $z(t) = -(l + \zeta)$

$$\text{so} \quad -\omega^2 m\cos\omega t - m\ddot{\zeta} = \gamma\zeta - mg$$

$$\Rightarrow \ddot{\zeta} + \frac{\gamma}{m}\zeta = g - \omega^2\cos\omega t = \hat{g}(t) \text{ say.}$$

(v) You solved ODEs similar to this in the first part of the course. Again

$$\zeta = \zeta_{CF} + \zeta_{PI}.$$

$$\xi_{CF} = A\cos\left(\sqrt{\frac{\gamma}{m}}\,t\right) + B\sin\left(\sqrt{\frac{\gamma}{m}}\,t\right) \quad \text{as before.}$$

For the particular integral let's try

$$\xi_{PI} = \frac{mg}{\gamma} + C.\cos\omega t \quad \text{for some } C.$$

Then $\ddot{\xi}_{PI} + \frac{\gamma}{m}\xi_{PI} = g - \omega^2\cos\omega t \Rightarrow$

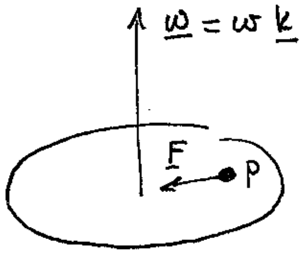
$$-\omega^2 C\cos\omega t + \frac{\gamma C}{m}\cos\omega t = -\omega^2\cos\omega t.$$

$$\text{So} \quad C = -\omega^2/(\gamma/m - \omega^2) = m\omega^2/(m\omega^2 - \gamma)$$

$$\xi = A\cos\left(\sqrt{\frac{\gamma}{m}}\,t\right) + B\sin\left(\sqrt{\frac{\gamma}{m}}\,t\right) + \frac{mg}{\gamma} + \frac{m\omega^2}{m\omega^2-\gamma}\cos\omega t.$$

for $\omega^2 \neq \gamma/m$.

Q2.



$$m\ddot{\underline{r}} = \underline{F} - 2m\underline{\omega}\wedge\dot{\underline{r}} - m\underline{\omega}\wedge(\underline{\omega}\wedge\underline{r})$$

$$\underline{r} = x(t)\underline{i} + y(t)\underline{j} + 0\underline{k} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

$$\text{So} \quad \underline{\omega}\wedge\dot{\underline{r}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \begin{pmatrix} -\omega\dot{y} \\ \omega\dot{x} \\ 0 \end{pmatrix}$$

Similarly,

$$\underline{\omega} \wedge \underline{r} = \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix} \quad \text{so} \quad \underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}) = \begin{pmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{pmatrix}$$

So

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{pmatrix} = \begin{pmatrix} -m\omega^2 x \\ -m\omega^2 y \\ 0 \end{pmatrix} - 2m \begin{pmatrix} -\omega \dot{y} \\ \omega \dot{x} \\ 0 \end{pmatrix} - m \begin{pmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{pmatrix}$$

$$\Rightarrow \cancel{m}\ddot{x} = 2\cancel{m}\omega \dot{y} \quad \& \quad \cancel{m}\ddot{y} = -2\cancel{m}\omega \dot{x}$$

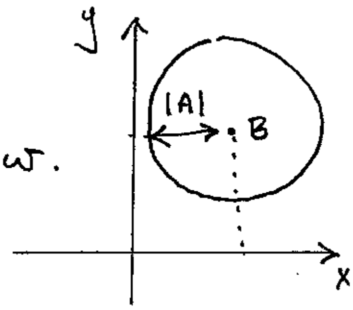
Let $\psi = x + iy$, then $\ddot{x} + i\ddot{y} = 2\omega \dot{y} - 2\omega i \dot{x}$

$$\Rightarrow \ddot{\psi} = -2i\omega \dot{\psi}, \quad \text{so} \quad \ddot{\psi} + 2i\omega \dot{\psi} = 0$$

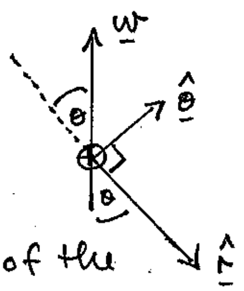
$\psi = A e^{-2i\omega t} + B$ for constants A & B (complex)

$$\Rightarrow x + iy = A(\cos 2\omega t - i \sin 2\omega t) + B$$

P moves around the circle with freq. 2ω.



Q3.



$\underline{k}$ out of the plane.

$$\underline{\omega} = -\omega \cos\theta \, \hat{\underline{r}} + \omega \sin\theta \, \hat{\underline{\theta}}.$$

$$\underline{r} = a\hat{\underline{r}}, \quad \dot{\underline{r}} = a\dot{\theta}\hat{\underline{\theta}}, \quad \ddot{\underline{r}} = a\ddot{\theta}\hat{\underline{\theta}} - a\dot{\theta}^2 \hat{\underline{r}}$$

$$m\ddot{\underline{r}} = \underline{F} - 2m\, \underline{\omega} \wedge \dot{\underline{r}} - m\, \underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}).$$

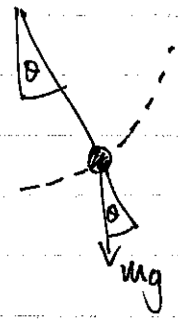
$$\underline{\omega} = \omega \begin{pmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad \text{so} \quad \underline{\omega} \wedge \dot{\underline{r}} = -\omega \begin{pmatrix} 0 \\ 0 \\ a\dot{\theta}\cos\theta \end{pmatrix}$$

ie. out of the plane.

Similarly,

$$\underline{\omega} \wedge \underline{r} = \omega \begin{pmatrix} 0 \\ 0 \\ -a\sin\theta \end{pmatrix} \quad \therefore \ \underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}) = \omega^2 \begin{pmatrix} -a\sin^2\theta \\ -a\sin\theta\cos\theta \\ 0 \end{pmatrix}.$$

So $m\begin{pmatrix} -a\dot\theta^2 \\ a\ddot\theta \\ 0 \end{pmatrix} = \begin{pmatrix} F_r \\ F_\theta \\ F_z \end{pmatrix} + 2m\omega\begin{pmatrix} 0 \\ 0 \\ a\dot\theta\cos\theta \end{pmatrix} - \omega^2 m\begin{pmatrix} -a\sin^2\theta \\ -a\sin\theta\cos\theta \\ 0 \end{pmatrix}$



The particle weight is $mg\cos\theta\,\hat{\underline{r}} - mg\sin\theta\,\hat{\underline{\theta}}$

And the reaction force is normal to the wire:

$$\underline{N} = \begin{pmatrix} N_r \\ 0 \\ N_z \end{pmatrix}, \text{ so } F_\theta = -mg\sin\theta.$$

So we can solve in the tangential direction:

$$ma\ddot\theta = -mg\sin\theta + \omega^2 ma\sin\theta\cos\theta.$$

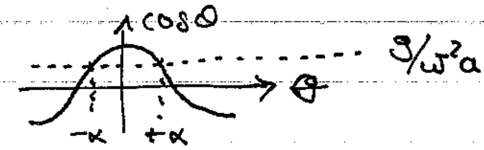
Note: The coriolis force is always perpendicular to the velocity $\Rightarrow$ here it only contributes to the normal reaction component N_z.

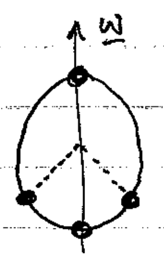
$\boxed{\text{(ii)}}$ For an equilibrium $\dot\theta = 0$ so $\cancel{m}g\sin\theta = \omega^2\cancel{m}a\sin\theta\cos\theta$

$\therefore\ \sin\theta = 0\ \ (\theta = 0 \text{ or } \theta = \pi)$

or $\cos\theta = g/\omega^2 a$ thus if $g/\omega^2 a < 1$ (ie. $\omega^2 > g/a$)

then we have two other solutions
$\theta = \pm\alpha,\ \alpha = \arccos(g/\omega^2 a)$.





$\}$ 4 possible equil. points.

$\boxed{\text{(iii)}}$ To test the stability: $\theta = \alpha + \varepsilon\psi(t)$

Then $ma\varepsilon\ddot\psi = -mg\sin(\alpha+\varepsilon\psi) + \omega^2 ma\sin(\alpha+\varepsilon\psi)\cos(\alpha+\varepsilon\psi)$

but $\varepsilon \ll 1$, and $\sin(\alpha+\varepsilon\psi)\cos(\alpha+\varepsilon\psi) = \frac{1}{2}\sin(2\alpha+2\varepsilon\psi)$

Taylor series: $\sin(\alpha+\varepsilon\psi) = \sin\alpha + \varepsilon\psi(t)\cos\alpha + \cdots$

$$\frac{1}{2}\sin(2\alpha+2\varepsilon\psi) = \frac{\sin 2\alpha}{2} + \frac{\cancel{2}\varepsilon\psi(t)\cos 2\alpha}{\cancel{2}} + \cdots$$

So, neglecting higher powers of ε leads to:

$$\varepsilon \cancel{m} a\ddot{\psi}(t) = -\cancel{m}g\sin\alpha + \omega^2 \cancel{m} a\sin\alpha\cos\alpha$$
$$+\,\varepsilon(-\cancel{m}g\cos\alpha + \omega^2\cancel{m}a\cos 2\alpha)\psi(t).$$

furthermore, we know that $\omega^2\sin\alpha\cos\alpha = g\sin\alpha$.

$$\Rightarrow a\ddot{\psi}(t) = (\omega^2 a\cos 2\alpha - g\cos\alpha)\psi.$$

So, if $\alpha = 0$ (bottom of the hoop)

$$\ddot{\psi} = \left(\omega^2 - \frac{g}{a}\right)\psi. \Rightarrow \ddot{\psi} + \left(\frac{g}{a} - \omega^2\right)\psi = 0.$$

solutions behave like $\exp\left\{i\left(\frac{g}{a} - \omega^2\right)^{1/2} t\right\}$

ie. oscillatory for $\omega^2 < g/a$, grow exponentially if $\omega^2 > g/a$.

The bottom of the hoop is stable up until $\omega^2 > g/a$, then unstable.

If $\alpha = \pi$

$$\ddot{\psi} = \left(\omega^2 + g/a\right)\psi$$

This always has exponentially growing solution

$\therefore$ the top of the hoop is always unstable.

At the two mirror image positions $\theta = \pm\alpha$ $(\omega^2 > g/a)$

$$\omega^2 a = g/\cos\alpha$$

So $\omega^2 a\cos 2\alpha - g\cos\alpha = g\left(\frac{\cos 2\alpha - \cos^2\alpha}{\cos\alpha}\right) = -g\frac{\sin^2\alpha}{\cos\alpha}$.

So $$\ddot{\psi} + \frac{g \sin^2\alpha}{a \cos\alpha} \psi = 0.$$

$\Rightarrow$ stable points with a frequency of oscillation of $\sqrt{\dfrac{g \sin^2\alpha}{a\cos\alpha}}$.

So on increasing ω from 0, the particle P starts at $\theta = 0$ ($\theta = \pi$ being unstable). When $\omega^2 > g/a$, $\theta = 0$ becomes unstable, but we then obtain 2 new equilibrium solutions at $\theta = \pm\alpha$ which remain stable. As ω is increased $\alpha \to \pm\pi/2$.